Vortex type equations and canonical metrics

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Abstract We introduce a notion of Gieseker stability for a filtered holomorphic vector bundle $\mathcal{F}$ over a projective manifold. We relate it to an analytic condition in terms of hermitian metrics on $\mathcal{F}$ coming from a construction of the Geometric Invariant Theory (G.I.T). We prove that if there is a $\boldsymbol{\tau}$-Hermite-Einstein metric h_{HE} on $\mathcal{F}$, then there exists a sequence of such balanced metrics that converges and its limit is h_{HE}. As a corollary, we obtain an approximation theorem for quiver Vortex equations and other classical equations.

Contents

1 Introduction

Intrinsic global methods have appeared in Kähler geometry with the fundamental work of E. Calabi and S-T. Yau in the study of Einstein type metrics. Recently, S.K. Donaldson in [16] has made a crucial advance in view of an algebro-geometric interpretation of the existence of Kähler-Einstein metrics by studying a notion of stability introduced by H. Luo for the couples $\left(M, L^k\right)$ where M is projective and L is a polarization over M. The main result of [16] (also see [6] for a survey) is the convergence of a sequence of 'balanced' metrics, constructed algebraically via the embeddings in the projective spaces $\mathbb{P}H^0\left(M, L^k\right)$ for k large, towards a constant scalar metric (when its existence is assumed *a priori*).
In the case of bundles (with or without some extra decorated structures), such correspondences have been proved in most of the known cases, at least when M is compact. Therefore, one can ask if the method of [16,14] can be applied to give an approximation of the equations that appear for these objects. In particular, this means that the gobal analysis used in Hitchin-Kobayashi-Donaldson-Uhlenbeck-Yau (we will say HKDUY in short) type correspondences contains some key technics for the construction of algebraic objects. By 'algebraic objects', we mean some canonical metrics coming from an underlying G.I.T construction in a finite dimensional framework. A preliminary work has been done by C. Drouet [17] for the case of Hermite-Einstein equations over a curve, and later X. Wang [38,39] has given a complete solution of the problem in any dimension using Gieseker's results and Donaldson's breakthrough.

In this paper, we are interested in a more general type of equation, so called Vortex equations, that appear in bosonic theories for non linear σ-models under the form of Bogomol'nyi-Prasad-Sommerfeld states. In the Kähler setting, and considering only a single bundle, these equations have been studied essentially by C.H. Taubes, S. Bradlow [8], O. García-Prada [18,20] and later D. Banfield [5] who proposed a generic frame. In [19] (see also [10]), O. García-Prada introduced a notion of coupled Vortex equations relative to a triple (i.e two holomorphic bundles and a bundle map between them). Later, this idea of considering Vortex equations for some data involving more than one bundle or sheaf has been developped in the work of L. Álvarez-Cónsul and O. García-Prada [1,3,2] who also investigated the related notion of quivers. Nevertheless, a G.I.T construction for the objects studied by Banfield or Álvarez-Cónsul and García-Prada is still to be achieved in full generality, even if some recent progress has been made in [21, 35]. From a general point of view, the study of Vortex equations and their moduli space of solutions has had some important consequences in algebraic geometry (for instance the Velinde formula and its generalizations) or in Gauge theory with the computation of Gromov-Witten invariants [33].

In our work, we will focus on the $\boldsymbol{\tau}$-Hermite-Einstein equations introduced by Álvarez-Cónsul and García-Prada [1] for a holomorphic filtration $\mathscr{F}$ (or filtered holomorphic bundle) over a smooth projective manifold. In a first part, we introduce a weaker notion of stability for holomorphic filtrations and we relate it to certain Gieseker spaces using a G.I.T construction (Theorem 1). We use a Kempf-Ness type argument to show that the stability condition can be written as an analytic condition involving the Bergman kernel associated to the filtration (Theorem 2). Thus, a sequence of canonical metrics will be defined, called balanced metrics as foreseen in [14] (Definition 12). Then we use an idea of Donaldson to find zeros of moment maps and we study the combined action of the Gauge group and the special unitary group $SU(N)$. Finally, we will prove in the fourth part (Theorem 5),

Theorem. Let M be a smooth projective manifold. If $\mathscr{F}$ is an irreducible holomorphic filtration of a holomorphic vector bundle $\mathcal{F}$ over M equipped with a metric h_{HE} solution of the $\boldsymbol{\tau}$-Hermite-Einstein equation

$$\sqrt{-1}\Lambda F_{h_{HE}} = \sum_i \widetilde{\tau}_i \pi^i_{h_{HE}}(\mathscr{F}),$$

then there exists a sequence of balanced metrics h_k on $\mathcal{F}$ which converges, up to conformal change, towards the metric h_{HE} in C^∞ topology.

As a corollary, we give a procedure to get an approximation of solutions of quiver Vortex equations (Theorem 7) using a dimensional reduction argument based on [1,2]. In particular, these equations overlap with Hermite-Einstein equations, special Witten triples (non abelian monopoles) and critical Higgs equations over a curve (see Section 5.3 for details).

2 Preliminaries

2.1 Notions of stability

The aim of this part is to introduce different notions of stability for a holomorphic filtration on a projective manifold. We will follow here the ideas of D. Gieseker and the Mumford theory in order to introduce a Gieseker space for which the G.I.T stable points correspond to stable holomorphic filtrations.

Let M be a projective manifold of complex dimension n, and L an ample line bundle on M.

Definition 1 *A filtration of sheaves of length m is a finite sequence of coherent subsheaves*

$$\mathscr{F} : 0 = \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow ... \hookrightarrow \mathcal{F}_m = \mathcal{F}$$

and we shall say that $\mathscr{F}$ is a holomorphic filtration if the sheaves $\mathcal{F}_i$ and $\mathcal{F}$ are subbundles.

Definition 2 *A subfiltration of the filtration* $\mathscr{F}$ *is a filtration of sheaves of length* m

$$\mathscr{F}' : 0 \hookrightarrow \mathcal{F}'_1 \hookrightarrow ... \hookrightarrow \mathcal{F}'_m = \mathcal{F}',$$

where $\mathcal{F}'$ *is a subsheaf of* $\mathcal{F}$ *and such that* $\mathcal{F}'_i \subseteq \mathcal{F}_i \cap \mathcal{F}'$ *for any* $1 \leq i \leq m$. *A subfiltration is said to be proper if* $r(\mathcal{F}') < r(\mathcal{F})$.

Definition 3 *A holomorphic filtration* $\mathscr{F}$ *is irreducible if it cannot be written as*

$$\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2,$$

where the $\mathscr{F}_i \neq \mathscr{F}$ *are holomorphic subfiltrations.*

Definition 4 *We will say that the filtration* $\mathscr{F}$ *is simple if any endomorphism* $f \in End(\mathcal{F})$ *which preserves the filtration (i.e.* $f(\mathcal{F}_i) \subset \mathcal{F}_i$*) is a scalar multiple of* Id.

We recall the notion of (slope) stability for a filtration (we refer to [1,9] for details).

Definition 5 *Let* $\mathscr{F}$ *be a filtration of length* m *and* $\boldsymbol{\tau} = (\tau_1, ..., \tau_{m-1})$ *a* $(m-1)$*-tuple of real numbers. Define the* $\boldsymbol{\tau}$*-degree of* $\mathscr{F}$ *as*

$$\deg_{\boldsymbol{\tau}}(\mathscr{F}) = \deg(\mathcal{F}) + \sum_{i=1}^{m-1} \tau_i r(\mathcal{F}_i),$$

and the $\boldsymbol{\tau}$*-slope of* $\mathscr{F}$ *as*

$$\mu_{\boldsymbol{\tau}}(\mathscr{F}) = \frac{\deg_{\boldsymbol{\tau}}(\mathscr{F})}{r(\mathcal{F})}.$$

We shall say that the filtration $\mathscr{F}$ *is* $\boldsymbol{\tau}$*-stable (resp. semi stable) if for all proper subfiltration* $\mathscr{F}' \hookrightarrow \mathscr{F}$, *we have*

$$\mu_{\boldsymbol{\tau}}(\mathscr{F}') < \mu_{\boldsymbol{\tau}}(\mathscr{F}) \qquad (resp. \ \leq).$$

A filtration is said to be polystable if it is a direct sum of $\boldsymbol{\tau}$*-stable filtrations with same slope* $\mu_{\boldsymbol{\tau}}$.

Remark 1 If the filtration reduces to a bundle $\mathcal{F}$ (i.e. $m = 1$) we recover the case of Mumford stability for bundles with $\tau_i = 0$ and in this case, we will denote by $\mu(\mathcal{F})$ the slope of $\mathcal{F}$. Notice that the $\boldsymbol{\tau}$-stability of a holomorphic filtration does not imply the Mumford stability of the subbundles.

Lemma 1 *Let* $\mathscr{F}^1$ *and* $\mathscr{F}^2$ *be two filtrations of torsion free sheaves with same length,* $\boldsymbol{\tau}$*-stable and of same slope. Let* $\varrho : \mathscr{F}^1 \to \mathscr{F}^2$ *a non zero homomorphism such that for all* i, $\varrho(\mathcal{F}^1_i) \subset \mathcal{F}^2_i$. *Then* ϱ *is injective. In particular, if a holomorphic filtration is stable then it is simple.*

Proof If ϱ is non injective then $\mathcal{F}^3 = \mathrm{Im}(\varrho)$ is a (proper) torsion free quotient of $\mathcal{F}^1$ and we get $\mathscr{F}^3$ subfiltration of $\mathscr{F}^2$ such that

$$\mu_{\boldsymbol{\tau}}(\mathscr{F}^3) > \mu_{\boldsymbol{\tau}}(\mathscr{F}^1) = \mu_{\boldsymbol{\tau}}(\mathscr{F}^2)$$

by Whitney product formula. Since $\mathscr{F}^2$ is stable and $\mathscr{F}^3 \subset \mathscr{F}^2$, one has necessarily that $r(\mathcal{F}^3) = r(\mathcal{F}^2)$.
Now it is also clear that for a holomorphic subfiltration $\mathscr{F}' \subset \mathscr{F}$ such that $r(\mathcal{F}) = r(\mathcal{F}')$, the following inequality always holds

$$\mu_{\boldsymbol{\tau}}(\mathscr{F}') \leq \mu_{\boldsymbol{\tau}}(\mathscr{F}). \tag{1}$$

This comes from the fact that if $\mathcal{F}' \ncong \mathcal{F}$ there exists an effective divisor D such that $\det(\mathcal{F}) \cong \det(\mathcal{F}' \otimes \mathcal{O}_M(D))$ and so $\mu(\mathcal{F}) = \mu(\mathcal{F}') + \frac{\deg(D)}{r(\mathcal{F}')} > \mu(\mathcal{F}')$. In the case of non holomorphic filtrations (i.e. given by torsion free sheaves), inequality (1) remains true. Then we get a contradiction: ϱ is injective.

If $\mathscr{F}^1$ and $\mathscr{F}^2$ are two holomorphic filtrations, then $\mathscr{F}^3$ is a holomorphic subfiltration of $\mathscr{F}^2$, with same slope and $r(\mathcal{F}^3) = r(\mathcal{F}^2)$. We notice that in the case of holomorphic filtrations, the only case of equality in (1) is $\mathscr{F}' = \mathscr{F}$. Thus, ϱ is an isomorphism. Therefore, if $\mathscr{F}$ is a holomorphic stable filtration, any non zero endomorphism ϱ of $\mathscr{F}$ such that $\varrho(\mathcal{F}_i) \subset \mathcal{F}_i$ is an isomorphism and by Schur lemma, $\{\varrho \in End(\mathcal{F}) : \varrho(\mathcal{F}_i) \subset \mathcal{F}_i\}$ is a divison algebra of finite dimension and isomorphic to $\mathbb{C}$. $\square$

Notation For a holomorphic filtration $\mathscr{F}$ of length m and a hermitian metric h on $\mathcal{F}$ correspond h-orthogonal smooth projections on the subbundle $\mathcal{F}_i$ of $\mathcal{F}$ for all $1 \leq i \leq m$, that we shall note

$$\pi^{\mathscr{F}}_{h,i} : \mathcal{F} \to \mathcal{F}_i$$

with the convention $\pi^{\mathscr{F}}_{h,m} = Id_{\mathcal{F}}$.

The main result of [1] is the existence of a HKDUY correspondence for holomorphic filtrations in terms of metrics of $\boldsymbol{\tau}$-Hermite-Einstein type.

Theorem 2.11 Fix ω a Kähler metric on the compact manifold M. Let $\boldsymbol{\tau} \in \mathbb{R}^{m-1}_+$ and $\mathscr{F}$ be a holomorphic filtration of length m. A holomorphic filtration $\mathscr{F}$ is $\boldsymbol{\tau}$-polystable if and only if there exists a smooth hermitian metric h solution of the equation

$$\sqrt{-1}\Lambda_\omega F_h + \sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h,i} = \mu_{\boldsymbol{\tau}}(\mathscr{F}) Id_{\mathcal{F}}. \tag{2}$$

This can be also written as

$$\sqrt{-1}\Lambda_\omega F_h = \sum_{i=1}^{m} \widetilde{\tau}_i \pi^i_h(\mathscr{F}), \tag{3}$$

where

$$\widetilde{\tau}_i = \mu_{\boldsymbol{\tau}}(\mathscr{F}) - \sum_{j=i}^{m-1} \tau_j, \qquad \widetilde{\tau}_m = \mu_{\boldsymbol{\tau}}(\mathscr{F}),$$

and $\pi_h^i(\mathscr{F}) = \pi_{h,i}^{\mathscr{F}} - \pi_{h,i-1}^{\mathscr{F}}$ is the projection (with respect to h) on the orthogonal of the subbundle $\mathcal{F}_{i-1}$ of $\mathcal{F}_i$ with the convention $\pi_h^1(\mathscr{F}) = \pi_{h,1}^{\mathscr{F}}$. We shall say under these conditions that h is $\boldsymbol{\tau}$-Hermite-Einstein. The filtration $\mathscr{F}$ will be said $\boldsymbol{\tau}$-Hermite-Einstein.

Remark 2 The assumption of non negativity of the real numbers τ_i is crucial in the proof of [1, Theorem 2.1]. Henceforth, when we consider the stability of a filtration of length m, it is relatively to a $(m-1)$-tuple of non negative real numbers. If we take the trace of (3), we notice that the parameters $\widetilde{\tau}_i$ satisfy the relation

$$\sum_{i=1}^{m} \widetilde{\tau}_i r(\mathcal{F}_i/\mathcal{F}_{i-1}) = \deg(\mathcal{F}),$$

which means that we only have $m-1$ degrees of freedom as expected.

We introduce now a notion of Gieseker stability for filtrations.

Definition 6 *Let $\mathbf{R} = (R_1, .., R_{m-1})$ be a collection of $(m-1)$ polynomials with rational coefficients of degrees $d_i < n$ and positive for k large. Define*

$$\mathscr{P}_{\mathbf{R},\mathscr{F}}(k) = \chi(\mathcal{F} \otimes L^k) + \sum_{i=1}^{m-1} r(\mathcal{F}_i) R_i(k)$$

the $\mathbf{R}$-Hilbert polynomial of the filtration $\mathscr{F}$ of length m. Then $\mathscr{F}$ is said to be Gieseker $\mathbf{R}$-stable (resp. semi-stable) if for k large, one has for all proper subfiltration $\mathscr{F}'$ of $\mathscr{F}$,

$$\frac{\mathscr{P}_{\mathbf{R},\mathscr{F}'}(k)}{r(\mathcal{F}')} < \frac{\mathscr{P}_{\mathbf{R},\mathscr{F}}(k)}{r(\mathcal{F})} \qquad (\textit{resp. } \leq).$$

We get immediately by applying Riemann-Roch formula,

Proposition 1 *If the filtration $\mathscr{F}$ is $\boldsymbol{\tau}$-Mumford stable, then it is also $\mathbf{R}$-Gieseker stable for*

$$R_i(k) = \tau_i k^{n-1} + O(k^{n-2}).$$

2.2 G.I.T construction and extended Gieseker space for filtrations

We present a G.I.T frame for the holomorphic filtrations on a projective manifold, inspired from the work of [25, 26, 35]. We introduce a 'Gieseker space' which parametrizes the Gieseker stable holomorphic filtrations. Firstly, we notice that the considered Gieseker semi-stable objects live in bounded families, i.e. are parametrized by a scheme of finite type over $\mathbb{C}$.

Proposition 2 *The set of isomorphy classes of Gieseker semi-stable filtrations given by torsion free coherent sheaves with fixed* $\mathbf{R}$*-Hilbert polynomial is bounded.*

Proof Let us consider a filtration $\mathscr{F}$. By considering the leading coefficients of $\mathscr{P}_{\mathbf{R},\mathcal{F}}(k)$ and using the Gieseker semi-stability condition, one reaches the conclusion that the slopes $\mu(\mathcal{F}_i)$ are all bounded. Now, we can apply the boundedness theorem of [31, Section 3]. □

Fix $\mathbf{R}$ a collection of $(m-1)$ polynomials as before and let $\mathscr{F}$ be a holomorphic filtration of length m and note $r_i = r(\mathcal{F}_i)$ with $r_0 = 0$, $r = r(\mathcal{F})$ and $\mathfrak{p}$ its $\mathbf{R}$-Hilbert polynomial. By Kodaira's embedding theorem, there exists an integer k_0 such that for $k \geq k_0$, the bundles $\mathcal{F}_i \otimes L^k$ are globally generated and the cohomology groups of higher dimension of $\mathcal{F}_i \otimes L^k$ are trivial, i.e.

$$H^j(M, \mathcal{F}_i \otimes L^k) = 0, \qquad \forall j \geq 1.$$

For such a k, consider a vector space V isomorphic to $H^0(M, \mathcal{F} \otimes L^k)$, and let (v_i) be a basis of V. One can construct a quasi-projective quot scheme $\mathfrak{Q}'$ parametrizing equivalence classes of quotients $\{q : V \otimes L^{-k} \to \mathfrak{F}\}$ where $\mathfrak{F}$ is a filtration of torsion free coherent sheaves of length m with $\mathbf{R}$-Hilbert polynomial equal to $\mathfrak{p}$ and $H^0(q \otimes id_{L^k})$ is an isomorphism. This gives us universal quotients $\tilde{q}_i : V \otimes \pi_M^*(L^{-k}) \to \tilde{\mathfrak{F}}_i$ over $\mathfrak{Q} \times M$ where $\mathfrak{Q}$ is the union of components of $\mathfrak{Q}'$ that contain $\mathbf{R}$-semi-stable elements. The line bundles $\det(\tilde{\mathfrak{F}}_i)$ induce morphisms $\upsilon_i : \mathfrak{Q} \to Pic(M)$ and we denote $\mathfrak{A}_i$ the union of the finitely many components of $Pic(M)$ hit by the morphism υ_i. By previous proposition, this does not depend on the choice of k and we can assume that k_0 is large enough so that for all $[\mathcal{L}] \in \mathfrak{A}_i$, $\mathcal{L} \otimes L^{(r-r_i)k}$ is globally generated and without higher cohomology. Fix a Poincaré line bundle $\tilde{\mathcal{L}}$ on $Pic(M) \times M$ and note $\tilde{\mathcal{L}}_{\mathfrak{A}_i}$ its restriction to $\mathfrak{A}_i \times M$. Again by previous proposition, we can assume that $\tilde{\mathcal{L}}_{\mathfrak{A}_i} \otimes \pi_M^* L^{(r-r_i)k}$ is globablly generated and without higher cohomology for $k \geq k_0$. Introduce the extended Gieseker space

$$\mathfrak{G} = \prod_{i=0}^{m-1} \mathbb{P}\left(Hom\left(\wedge^{r-r_i} V \otimes \mathcal{O}_{\mathfrak{A}_i}, (\pi_{\mathfrak{A}_i})_*(\tilde{\mathcal{L}}_{\mathfrak{A}_i} \otimes \pi_M^* L^{(r-r_i)k}) \right)^\vee \right).$$

The morphism $(\pi_{\mathfrak{Q}})_*(\wedge^{r-r_i}(\tilde{q}_i \otimes id_{\pi_M^* L^k}) : \wedge^{r-r_i} V \otimes \mathcal{O}_{\mathfrak{Q}} \to (\pi_{\mathfrak{Q}})_*(\det(\tilde{\mathfrak{F}}_i) \otimes \pi_M^* L^{(r-r_i)k})$ gives rise to an injective and $SL(V)$-equivariant morphism

$$Gies : \mathfrak{Q} \to \mathfrak{G}.$$

Moreover, the Gieseker space $\mathfrak{G}$ maps $SL(V)$-invariantly to $\prod_i \mathfrak{A}_i$ and the fibers are close $SL(V)$-invariant subschemes; i.e., over the point $(\mathcal{L}_1, .., \mathcal{L}_m) \in \prod_i \mathfrak{A}_i$, sits the space

$$\widetilde{\mathfrak{G}}_{(\mathcal{L}_1,..,\mathcal{L}_m)} = \prod_i \mathbb{P}\left(Hom\left(\wedge^{r-r_i} V, H^0(\mathcal{L}_i \otimes L^{(r-r_i)k}) \right)^\vee \right).$$

We now focus on determining the (semi-)stable points of this space. For the holomorphic filtration $\mathscr{F}$ of lentgh m, let π_i be the natural onto map from $\mathcal{F}$ to the quotient $\mathcal{F} \otimes$

$L^k/\mathcal{F}_i \otimes L^k$ with the convention $\pi_0 = Id$. To the filtration $\mathscr{F}$, we associate by the following morphisms of evaluation,

$$T_i : (v_{j_1} \wedge ... \wedge v_{j_{r-r_i}}) \mapsto \Big(p \mapsto \pi_i ev_p(v_{j_1}) \wedge ... \wedge \pi_i ev_p(v_{j_{r-r_i}}) \Big),$$

a point in the space

$$\widetilde{\mathfrak{G}}_k = \prod_{i=0}^{m-1} \mathbb{P}Hom \left(\wedge^{r-r_i} V, H^0 \left(M, \det(\mathcal{F} \otimes L^k/\mathcal{F}_i \otimes L^k) \right) \right).$$

The action of $SL(V)$ is given by

$$g \star (T_0, ..., T_{m-1}) = \left(T_0 \circ \wedge^r g^{-1}, ..., T_{m-1} \circ \wedge^{r-r_{m-1}} g^{-1} \right),$$

and we can consider for a choice of parameters $\varepsilon_i > 0$ the G.I.T stability of a point of $\widetilde{\mathfrak{G}}_k$ relatively to a $SL(V)$-linearization of the very ample bundle $\mathcal{O}_{\widetilde{\mathfrak{G}}_k}(\varepsilon_0, ..., \varepsilon_{m-1})$.

Let $\underline{\lambda} : G_t \to SL(V)$ be a 1-parameter subgroup and v_i a basis of V such that G_t acts on V by $\underline{\lambda}$ with the weights $\gamma_i \in \mathbb{Z}$, i.e. for all $t \in G_t$,

$$\underline{\lambda}(t) \cdot v_i = t^{\gamma_i} v_i.$$

Of course, it can be assumed that $\gamma_i \leq \gamma_{i+1}$ and $\sum_i \gamma_i = 0$. For any multi-index $I = (i_1, ..., i_{r-r_j})$ of length $|I| = r - r_j$ with $1 \leq i_1 < ... < i_{r-r_j} \leq \dim(V)$, let $v_I = v_{i_1} \wedge ... \wedge v_{i_{r-r_j}}$ and $\gamma_I = \gamma_{i_1} + ... + \gamma_{i_{r-r_j}}$. The group $SL(V)$ acts on $\bigwedge^{r-r_j} V$ with weight γ_I relatively to the basis v_I. The classical Hilbert-Mumford criterion asserts that the point of $\widetilde{\mathfrak{G}}_k$ is G.I.T-(semi-)stable relatively to the linearization that we have fixed and the action of $SL(V)$ if and only if, for all 1-parameter subgroup,

$$\sum_{j=0}^{m-1} \varepsilon_j \min_{\{I : |I| = r - r_j\}} \{\gamma_I : T_j(v_I) \neq 0\} < 0 \qquad (\text{resp. } \leq).$$

Remark 3 With Riemann-Roch theorem, we compute the dimension of V,

$$\begin{aligned}\dim(V) &= \int_M Ch(\mathcal{F} \otimes L^k) Todd(M) \\ &= r(\mathcal{F}) k^n \int_M \frac{c_1(L)^n}{n!} + k^{n-1} \int_M \left(\frac{r}{2} c_1(M) + c_1(\mathcal{F}) \right) \frac{c_1(L)^{n-1}}{(n-1)!} + ...\end{aligned}$$

Consider now the case of a subspace $V' \subset V$ and the action of the subgroup associated is given by

$$\lambda(t) = \begin{pmatrix} t^{-\mathrm{codim}(V')} & & 0 \\ & \ddots & \\ 0 & & t^{\dim(V')} \end{pmatrix},$$

with $\gamma_1 = .. = \gamma_{\dim(V')} = -\mathrm{codim}(V')$ and $\gamma_{\dim(V')+1} = .. = \gamma_{\dim(V)} = \dim(V')$. Via the morphism $V \otimes \mathcal{O}_M \to \mathcal{F} \otimes L^k$ obtained for k sufficiently large, we have a holomorphic filtration

$$\begin{aligned} 0 \subsetneq \mathcal{F}_{(V')} \subset \mathcal{F}_{(V)} = \mathcal{F}, \\ \mathcal{F}_{(V'),i} = \mathcal{F}_i \cap \mathcal{F}_{(V')} \end{aligned}$$

and under these conditions,

$$\begin{aligned} \min_{\{|I|=r-r_j\}} \{\gamma_I : T_j(v_I) \neq 0\} = \dim(V') \left(r(\mathcal{F}) - r(\mathcal{F}_i) \right) \\ - \dim(V) \left(r(\mathcal{F}_{(V')}) - r(\mathcal{F}_{(V'),i}) \right). \end{aligned}$$

Thus, we have shown that if the point of $\widetilde{\mathfrak{G}}_k$, defined by the filtration $\mathscr{F}$, is G.I.T (semi-)stable, then we have

$$\begin{aligned} &\varepsilon \left(\dim(V') r(\mathcal{F}) - \dim(V) r(\mathcal{F}_{(V')}) \right) \\ &\quad + \textstyle\sum_{i=0}^{m-1} \varepsilon_i \left(\dim(V) r(\mathcal{F}_{(V'),i}) - \dim(V') r(\mathcal{F}_i) \right) < 0, \end{aligned}$$

where we have set

$$\varepsilon = \sum_{i=0}^{m-1} \varepsilon_i .$$

In fact, we also have the converse:

Lemma 2 *The point of $\widetilde{\mathfrak{G}}_k$ defined by the holomorphic filtration $\mathcal{F}$ is G.I.T (semi-)stable if and only if for any subspace $V' \subset V$,*

$$\begin{aligned} &\varepsilon \left(\dim(V') r(\mathcal{F}) - \dim(V) r(\mathcal{F}_{(V')}) \right) \\ &\quad + \textstyle\sum_{i=0}^{m-1} \varepsilon_i \left(\dim(V) r(\mathcal{F}_{(V'),i}) - \dim(V') r(\mathcal{F}_i) \right) < 0, \end{aligned}$$

where $\mathscr{F}_{(V')}$ is the filtration generated by $V' \otimes \mathcal{O}_M$ and $\mathcal{F}/\mathcal{F}_{(V')}$ is torsion free.

Proof Let $(v_1, ..., v_{\dim(V)})$ be a basis of V. If one denotes $\mathcal{F}_{(i)} = \mathcal{F}_{\langle v_1, ..., v_i \rangle}$, we have a filtration

$$\mathcal{F}_{(0)} \subset ... \subset \mathcal{F}_{\dim(V)} = \mathcal{F}$$

and we get $\mathcal{F}_{(i)} = \mathcal{F}_{(i-1)}$ or $r(\mathcal{F}_{(i)}) > r(\mathcal{F}_{(i-1)})$. Consequently, there exist r integers between 1 and $\dim(V)$ that mark the jumps of ranks. We shall denote them $(k_1, .., k_r)$. If one considers the action associated to (v_i, γ_i), we get by [26, Lemma 1.23], $\min_I \{\gamma_I : T_0(v_I) \neq 0\} = \gamma_{k_1} + ... + \gamma_{k_r}$. From a similar way, there exist $r - r_j$ integers between 1 and $\dim(V)$ that mark the jumps of ranks for $\mathcal{F} \otimes L^k / \mathcal{F}_{(i),j} \otimes L^k$. We denote them $(k_1^j, .., k_{r-r_j}^j)$. Therefore, we get

$$\min_I \{\gamma_I : T_j(v_I) \neq 0\} = \gamma_{k_1^j} + ... + \gamma_{k^j_{r-r_j}} .$$

In order to apply the Hilbert-Mumford criterion, we shall consider the set of all the vectors with weight γ_i. Of course, these vectors are generated by the following vectors

$$\gamma^{(i)} = (\underbrace{i-\dim(V),\dots,i-\dim(V)}_{i},\ \underbrace{i,\dots,i}_{\dim(V)-i})$$

for $i = 1,...,\dim(V)$. All weighted vectors $\gamma = \left(\gamma_1,...,\gamma_{\dim(V)}\right)$ can be expressed as $\gamma = \sum_{i=1}^{\dim(V)} c_i\gamma^{(i)}$ where $c_i = \frac{\gamma_{i+1}-\gamma_i}{\dim(V)}$ are non negative rational coefficients. Let us apply the Hilbert-Mumford criterion to $\gamma^{(i)}$; we get

$$\begin{aligned}\mu^{(i)} &= \textstyle\sum_{j=0}^{m-1}\varepsilon_j \min_{\{I:|I|=r-r_j\}}\{\gamma_I : T_j(v_I)\neq 0\}\\ &= -\dim(V)\left(\textstyle\sum_j\varepsilon_j \max_l\{k_l^j \leq i\}\right) + i\left(\textstyle\sum_j (r-r_j)\varepsilon_j\right).\end{aligned}$$

If i grows, $\mu^{(i)}$ goes down except for k_j or one k_j^l. One has to evaluate $\mu^{(i)}$ at the values k_j-1 or k_j^l-1. This leads to

$$\mu^{(i)} = -\dim(V)\textstyle\sum_l\varepsilon_l(j^l-1) + \sum_l\varepsilon_l(r-r_l)(k_j^l-1).$$

Eventually, we can forget the choice of the basis v_i and we get that the point of $\widetilde{\mathfrak{G}}_k$ is G.I.T stable (resp. semi-stable) if and only if

$$\begin{aligned}\varepsilon\dim(V')r(\mathcal{F}) - \dim(V')\textstyle\sum_i\varepsilon_i r(\mathcal{F}_i) < \varepsilon\dim(V)r(\mathcal{F}_{(V')})&\\ - \dim(V)\textstyle\sum_i\varepsilon_i r(\mathcal{F}_{(V'),i})&\end{aligned}$$

(resp. $\leq$) for any subspace $0\neq V'\subset V$, with $r\left(\mathcal{F}_{(V')}\right)\leq r(\mathcal{F})$. □

By [26, Lemma 1.26], the G.I.T stability (resp. semi-stability) can be written as a condition on the subsheaves of $\mathcal{F}$ instead of the subspaces of V,

$$\begin{aligned}&\dim(V\cap H^0(\mathcal{F}'\otimes L^k))\left(\varepsilon r(\mathcal{F}) - \textstyle\sum_i\varepsilon_i r(\mathcal{F}_i)\right)\\ &\qquad < \dim(V)\left(\varepsilon r(\mathcal{F}') - \textstyle\sum_i\varepsilon_i r\left(\mathcal{F}_{V\cap H^0(\mathcal{F}'\otimes L^k),i}\right)\right)\end{aligned}$$

(resp. $\leq$) for any proper holomorphic filtration $\mathscr{F}'\subset\mathscr{F}$. Indeed, if $\mathcal{F}'\subset\mathcal{F}$, choose $V' = H^0(\mathcal{F}'\otimes L^k)\cap V$. Then for the morphism $q: V\otimes\mathcal{O}_M\to\mathcal{F}\otimes L^k$, $q(V'\otimes\mathcal{O}_M)\subset\mathcal{F}'\otimes L^k$ and $r(\mathcal{F}') = r(\mathcal{F}_{(V')})$. For the converse, we consider $V'\subset V$ and set $\mathcal{F}' = q(V'\otimes\mathcal{O}_M)$. Thus, we have $V'\subset V\cap H^0(\mathcal{F}'\otimes L^k)$ and $r(\mathcal{F}_{(V')}) = r(\mathcal{F}')$.

Now, it is clear that the Gieseker stability condition for the holomorphic filtration $\mathcal{F}$ implies the previous condition for a convenient choice of $\{\varepsilon,\varepsilon_1,...,\varepsilon_{m-1}\}$, i.e. we have proved the

Theorem 1 *Let* $\mathbf{R} = (R_1, ..., R_{m-1})$ *be a collection of* $(m-1)$ *polynomials with rational coefficients of degree* $d_i < n$ *and with positive leading coefficient. The holomorphic filtration* $\mathscr{F}$ *of length* m *is* $\mathbf{R}$*-stable (resp. semi-stable) if for* k *large, the associated point of* $\widetilde{\mathfrak{G}}_k$ *is G.I.T stable (resp. semi-stable) respectively to the polarization* $\mathcal{O}_{\widetilde{\mathfrak{G}}_k}(\varepsilon_0, ..., \varepsilon_{m-1})$ *and the action of* $SL(V)$ *where we have fixed*

$$\varepsilon_0 = 1 - \sum_{i=1}^{m-1} \frac{R_i(k)}{k^n},$$
$$\varepsilon_i = \frac{R_i(k)}{k^n} \qquad (1 \leq i \leq m-1).$$

Remark 4 This G.I.T construction extends the previous work of [13] in which a G.I.T construction is given for one step filtrations and a similar result is proved.

3 G.I.T stability and balanced metrics

In this part, we apply the Kempf-Ness criterion to the Gieseker spaces that we have just constructed for holomorphic filtrations. This means that the condition of G.I.T stability can be transposed as the existence of a certain sequence of ‘balanced’ metrics defined on the finite dimension vector space $H^0(\mathcal{F} \otimes L^k)$ that are critical points of certain functionals of Kempf-Ness type.
The balance condition for applications was conceptualized by S.K. Donaldson [14] in the following way. Let’s assume that we are given the following objects: a holomorphic map $f : \Xi \to W$ where (Ξ, ω_0) is compact Kähler and W is a vector space of finite dimension embedded by π^W as a co-adjoint orbit in $Lie(G)^*$ where G is reductive linear. Then, the center of mass of f in $Lie(G)^*$ is given by

$$\int_{Lie(G)^*} \pi_*^W \left(f_* \left(\frac{\omega_0^n}{n!} \right) \right).$$

Under this setting, f is said to be balanced if the orbit of f under the action of $Lie(G)^*$ contains a center of mass null. This means that we demand that the moment map defined by integration on $C^\infty(\Xi, W)$ respectively to the action of G admits a zero in the complex orbit of f.

In all the following, our polarization L will be equipped with a smooth hermitian metric h_L such that the curvature $c_1(L, h_L)$ is a Kähler metric on M that we denote ω, which means that

$$\omega = -\frac{i}{2\pi} \partial\overline{\partial} \log(h_L).$$

Let $dV = \frac{\omega^n}{n!}$ be the volume form relatively to ω.

Notation We define $Met(\Upsilon)$ the space of smooth hermitian metrics for the vector space or the vector bundle Υ. Let F be a hermitian vector bundle on (M,ω). We associate to a metric $h \in Met(F)$ the L^2 hilbertian metric on $H^0(M,F)$

$$Hilb_\omega(h) = \int_M \langle .,. \rangle_h \frac{\omega^n}{n!} \in Met(H^0(M,F)).$$

We will also need the well-known fact:

Definition 7 *If V_1 and V_2 are two vector spaces of finite dimension N_1, N_2 equipped with metrics h_1, h_2, and $T : V_1 \to V_2$ is a linear map, then the Hilbert-Schmidt norm $||T||_{h_1,h_2}$ can be computed using any othonormal basis $(v_i^1)_{i=1,..,N_1}$ of V_1 by $||T||^2_{h_1,h_2} = \sum_{i=1}^{N_1} |T(v_i^1)|^2_{h_2}$.*

3.1 Gieseker stable holomorphic filtrations and Kempf-Ness type functionals

We fix a holomorphic filtration $\mathscr{F}$ of length m and for k large, we set $N = h^0\left(M, \mathcal{F}\otimes L^k\right)$ and π_i the natural surjection onto $\mathcal{F}\otimes L^k/\mathcal{F}_i\otimes L^k$ with the convention that $\pi_0 = Id$. Let $\widetilde{\mathbf{Z}}_{\mathbf{ss}} \subset \widetilde{\mathfrak{G}}_k$ be the open scheme of G.I.T semi-stable points with respect to the linearization $\mathcal{O}_{\widetilde{\mathfrak{G}}_k}(\varepsilon_0,..,\varepsilon_{m-1})$ where the constants ε_i are fixed by Theorem 1.

Fix a metric H on the vector space $H^0(\mathcal{F}\otimes L^k)$. We have a quotient metric on the hermitian bundle $\mathcal{F}\otimes L^k$ induced by $V \twoheadrightarrow \mathcal{F}\otimes L^k$ and consequently a metric $||.||$ on $\Lambda^{r-r_i}(\mathcal{F}\otimes L^k)$. By the isomorphism $i : V \tilde{\to} H^0(M,\mathcal{F}\otimes L^k)$, we get a metric $h_{\widetilde{\mathbf{V}}_{\mathbf{k},\mathbf{i}}}$ on the space $\widetilde{\mathbf{V}}_{k,i} = Hom(\wedge^{r-r_i}V, H^0(\det(\mathcal{F}\otimes L^k/\mathcal{F}_i\otimes L^k)))$. We set

$$\widetilde{\mathbf{V}}_k = \widetilde{\mathbf{V}}_{k,0} \times ... \times \widetilde{\mathbf{V}}_{k,m-1},$$

and get a natural metric $|||.,.|||_{\widetilde{\mathbf{V}}_k} = h^{\varepsilon_0}_{\widetilde{\mathbf{V}}_{k,0}} \times ... \times h^{\varepsilon_{m-1}}_{\widetilde{\mathbf{V}}_{k,m-1}}$. Let $\mathbf{z}$ be a point in $\widetilde{\mathbf{Z}}_{\mathbf{ss}}$ and $\tilde{\mathbf{z}} \in \mathcal{O}_{\widetilde{\mathfrak{G}}_k}(-\varepsilon_0,...,-\varepsilon_{m-1})_{\mathbf{z}}$ be a lifting. We can evaluate the metric $|||.,.|||_{\widetilde{\mathbf{V}}_k}$ at that point,

$$|||\tilde{\mathbf{z}}|||^2_{\widetilde{\mathbf{V}}_k} = C(i)\prod_{j=0}^{m-1}\left(\int_M \sum_{\substack{1\le i_1<\dots \\ \dots<i_{r-r_j}\le N}} \left\|\pi_j\circ s_{i_1}(p)\wedge\dots\wedge\pi_j\circ s_{i_{r-r_j}}(p)\right\|^2 dV(p)\right)^{\varepsilon_j}$$

where $(s_i)_{i=1,..,N}$ is an H-orthonormal basis of $H^0\left(M,\mathcal{F}\otimes L^k\right)$, $C(i) > 0$ is a constant depending only on the isomorphism i.

Remark 5 Our construction does not depend on the choice of the metric on the determinant bundles $\det(\mathcal{F}\otimes L^k/\mathcal{F}_i\otimes L^k)$.

Definition 8 *We define the functional for $g \in SL(V)$,*

$$\widetilde{F_{k,\mathscr{F}}}(g)=\sum_{j=0}^{m-1} \varepsilon_j \log \int_M \frac{\sum_{\substack{1\leq i_1<\ldots\\ \ldots<i_{r-r_j}\leq N}} \left\|\pi_j(g\cdot s_{i_1}) \wedge \ldots \wedge \pi_j(g\cdot s_{i_{r-r_j}})\right\|^2}{\sum_{\substack{1\leq i_1<\ldots\\ \ldots<i_{r-r_j}\leq N}} \left\|\pi_j s_{i_1} \wedge \ldots \wedge \pi_j s_{i_{r-r_j}}\right\|^2} dV.$$

We can sum up our situation by the following lemma:

Lemma 3 *The following conditions are equivalent:*
– The holomorphic filtration $\mathscr{F}$ is $\mathbf{R}$-Gieseker polystable for a collection $\mathbf{R} = (R_1, ..., R_{m-1})$ of rational polynomials of degree $n-1$ with positive leading coefficient.
– There exists an integer k_0 such that for all $k \geq k_0$, the functionals $\widetilde{F_{k,\mathscr{F}}} : SL(V) \to \mathbb{R}$ admit a positive minimum where it has been assumed that

$$\varepsilon_i = \frac{R_i(k)}{k^n}, \qquad \varepsilon_0 = 1 - \sum_{i=1}^{m-1} \frac{R_i(k)}{k^n}$$

for all $i = 1, ..., m-1$.

Remark 6 If the holomorphic filtration $\mathscr{F}$ is $\boldsymbol{\tau}$-Mumford stable then there exists an integer k_0 such that for all $k \geq k_0$, the functionals $\widetilde{F_{k,\mathscr{F}}}$ admit a positive minimum and are proper, under the assumption of $\varepsilon_0 = 1 - \sum_{i=1}^{m-1} \frac{\tau_i}{k}$ and $\varepsilon_i = \frac{\tau_i}{k}$.

Note that we can also consider $\widetilde{F_{k,\mathscr{F}}}$ as a functional on the space $Met(V) \times SL(V)$, i.e. on a finite dimensional space. We are going to see that we can relate to the functional $\widetilde{F_{k,\mathscr{F}}}$ another functional, this time on the infinite dimensional space $Met(\mathcal{F}) \times SL(V)$. This motivates the following definition and theorem (see [34, Section 3] for details).

Definition 9 *Consider a Kähler manifold (Ξ, ω) and a moment map μ associated to the action of a compact linear group Γ such that $\Gamma^{\mathbb{C}}$ acts holomorphically. To the moment map μ corresponds a functional*

$$I_\mu : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$$

called the 'integral of the moment map μ' and which satisfies the properties:
– For all $p \in \Xi$, the critical points of the restriction $I_{\mu,p}$ of I_μ to $\{p\} \times \Gamma^{\mathbb{C}}$ coincide with the points in the orbit $\Gamma^{\mathbb{C}} p$ for which the moment map vanishes;
– The restriction $I_{\mu,p}$ to the lines $\{e^{\lambda u} : u \in \mathbb{R}\}$ where $\lambda \in Lie\left(\Gamma^{\mathbb{C}}\right)$ is convex.

Theorem 3.11 There exists a unique application $I_\mu : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$ which satisfies the two properties:
1. $I_\mu(p, e) = 0$ for all $p \in \Xi$;
2. $\frac{d}{du} I_\mu\left(p, e^{i\lambda u}\right)_{|u=0} = \langle \mu(p), \lambda \rangle$ for all $\lambda \in Lie(\Gamma)$.

Now, for k sufficiently large, we have the embeddings of M into the Grassmanians of $r-r_j$ quotients defined by:

$$i_{k,j}: \begin{array}{l} M \hookrightarrow Gr\left(N, r-r_j\right) \\ p \ \mapsto \ker\left(\pi_j \circ ev_p : V \to \mathcal{F}\otimes L^k/\mathcal{F}_j \otimes L^k|_p\right)^{\vee}. \end{array} \tag{4}$$

Let $\underline{\mathbf{U}}_{N,r}$ be the universal bundle on the Grassmannian of r-quotients of the Grassmannian $Gr(N,r)$. We denote $\mathbf{\Pi} = \prod_{i=0}^{m-1} Gr(N, r-r_i)$ and $\pi_{Gr,i} : \mathbf{\Pi} \to Gr(N, r-r_i)$ for $i = 0, ..., m-1$ the natural projections. We lift the Fubini-Study metrics on each factor $Gr(N, r-r_i)$ with weight ε_i. This induces a symplectic metric on $C^\infty(M, \mathbf{\Pi})$,

$$\Omega_{(i_{k,*})}(\overrightarrow{x}, \overrightarrow{y}) = \sum_{i=0}^{m-1} \int_M \varepsilon_i \pi^*_{Gr,i} \omega_{FS}(\overrightarrow{x}, \overrightarrow{y}) dV,$$

where $\overrightarrow{x}, \overrightarrow{y} \in C^\infty(M, (i_{k,0}, ..., i_{k,m-1})^* T\mathbf{\Pi})$. The moment map associated to $\Omega_{(i_{k,*})}$ for the action of the special unitary group $SU(N)$ on $C^\infty(M, \mathbf{\Pi})$ is

$$\begin{aligned} \mu_{\mathscr{F},k}(i_{k,*}) &= \int_M \sum_{j=0}^{m-1} \varepsilon_j \mathsf{Q}_j {}^{\mathbf{t}}\overline{\mathsf{Q}_j} dV - V \frac{\sum_{j=0}^{m-1}(r-r_j)\varepsilon_j}{N} Id, \\ &= \int_M (\varepsilon_0 + \sum_{j=1}^{m-1} \varepsilon_j) \mathsf{Q}_0 {}^{\mathbf{t}}\overline{\mathsf{Q}_0} dV - \int_M \sum_{j=1}^{m-1} \varepsilon_j \left(\mathsf{Q}_0 {}^{\mathbf{t}}\overline{\mathsf{Q}_0} - \mathsf{Q}_j {}^{\mathbf{t}}\overline{\mathsf{Q}_j}\right) dV \\ &\quad -V \frac{r - \sum_{j=0}^{m-1} r_j \varepsilon_j}{N} Id, \end{aligned}$$

where $[\mathsf{Q}_j]$ represents a point of $Gr(N, r-r_j)$ i.e. $\mathsf{Q}_j : \mathcal{F}\otimes L^k/\mathcal{F}_j \otimes L^k_{|p} \to V$ is an isometry respectively to h and H, and represents the matrix of the endomorphism $\pi_j \circ ev_p$ expressed in an orthonormal basis of $\ker(\pi_j \circ ev_p)^\perp$ and in an orthonormal basis of V.
Set $\mathbf{U}_{r,N} = \pi^*_{Gr,0} \underline{\mathbf{U}}_{N,r}$. Since $\mathcal{F}\otimes L^k \simeq j_k^* \mathbf{U}_{r,N}$ where $j_k : M \hookrightarrow \mathbf{\Pi}$ is induced by the maps $i_{k,l}$ and π_i, we get a new smooth hermitian metric on $\mathcal{F}\otimes L^k$ associated to the metric H on $H^0(M, \mathcal{F}\otimes L^k)$.

Definition 10 *Let $FS_k = FS_k(H) \in Met(\mathcal{F}\otimes L^k)$ be the hermitian metric on $\mathcal{F}\otimes L^k$ induced by*

$$\langle .,. \rangle_{FS_k} = \left\langle \frac{N}{Vr - V\sum_{j=1}^{m-1} \varepsilon_j r_j} \left(Id_{\mathcal{F}} - \sum\nolimits_{j=1}^{m-1} \varepsilon_j \pi^{\mathscr{F}}_{h,j} \right) .,. \right\rangle_h \tag{5}$$

where h is the quotient metric on $\mathcal{F}\otimes L^k$ induced by H.

Remark 7 This last expression is well defined since we get for k sufficiently large, $0 < \sum_{j=1}^{m-1} \varepsilon_j < 1$.

Definition 11 *Define the functional on $SL(V)$,*

$$\widetilde{KN_{k,\mathscr{F}}}(g)=\sum_{j=0}^{m-1}\frac{\varepsilon_j}{2}\int_M \log\frac{\sum_{\substack{1\leq i_1<\dots\\ \dots<i_{r-r_j}\leq N}}\left\|\pi_j(g\cdot s_{i_1})\wedge\dots\wedge\pi_j(g\cdot s_{i_{r_j}})\right\|^2}{\sum_{\substack{1\leq i_1<\dots\\ \dots<i_{r-r_j}\leq N}}\left\|\pi_j s_{i_1}\wedge\dots\wedge\pi_j s_{i_{r_j}}\right\|^2}dV$$

where $(s_i)_{i=1,..,N}$ is any H-orthonormal basis of $H^0\left(M,\mathcal{F}\otimes L^k\right)$.

Lemma 4 $\widetilde{KN_{k,\mathscr{F}}}$ *is the integral of the moment map $\mu_{\mathscr{F},k}$.*

Proof From [32], we know that a potential of the Fubini-Study metric at the point $[\mathsf{Q}]$ in the Grassmannian $Gr(N,r)$ is

$$\log\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{Q}\right).$$

In order to prove the lemma, we simply need to check that for any trace free matrix S,

$$\frac{d}{du}\left(\widetilde{KN_{k,\mathscr{F}}}\left(ge^{\mathsf{S}u}\right)\right)_{|u=0}=\mu_{\mathscr{F},k}(g)\in SL(N).$$

Let $[\mathsf{Q}_0(p)]$ represent the point in the Grassmannian $Gr(N,r)$ given by the embedding defined by (4), at $p\in M$. Therefore, we obtain up to a modification of the matrix $\mathsf{Q}_0(p)$ (i.e by considering instead the unitary matrix $\mathsf{Q}_0\left({}^{t}\overline{\mathsf{Q}_0}\mathsf{Q}_0\right)^{-1/2}$),

$$\int_M\log\left(\frac{\sum_{\substack{1\leq i_1<\dots\\ \dots<i_{r_j}\leq N}}\|g\cdot s_{i_1}\wedge\dots\wedge g\cdot s_{i_r}\|^2}{\sum_{\substack{1\leq i_1<\dots\\ \dots<i_{r_j}\leq N}}\|s_{i_1}\wedge\dots\wedge s_{i_r}\|^2}\right)dV=\int_M\log\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}_0}\,{}^{\mathsf{t}}\overline{g}g\mathsf{Q}_0\right)dV$$

with ${}^{\mathsf{t}}\overline{\mathsf{Q}_0(p)}\mathsf{Q}_0(p)=Id_{r\times r}$.
From a similar way, let $[\mathsf{Q}_i]$ be the point of $Gr(N,r-r_i)$ induced by $\pi_i\circ ev_p$. Thus,

$$\widetilde{KN_{k,\mathscr{F}}}(g)=\frac{1}{2}\left(\sum_{j=0}^{m-1}\varepsilon_j\int_M\log\frac{\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}_j}\,{}^{\mathsf{t}}\overline{g}g\mathsf{Q}_j\right)}{\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}_j}\mathsf{Q}_j\right)}dV\right).$$

Therefore, we get that for any trace free matrix S and $g\in SU(N)$,

$$\frac{d}{du}\left(\widetilde{KN_{k,\mathscr{F}}}\left(e^{\mathsf{S}u}\right)\right)=\sum_{j=0}^{m-1}\frac{\varepsilon_j}{2}\int_M\mathrm{tr}\left(\left({}^{\mathsf{t}}\overline{\mathsf{Q}_j}\,e^{{}^{\mathsf{t}}\overline{\mathsf{S}}u}e^{\mathsf{S}u}\mathsf{Q}_j\right)^{-1}\left({}^{\mathsf{t}}\overline{\mathsf{Q}_j}e^{{}^{\mathsf{t}}\overline{\mathsf{S}}u}\left(\mathsf{S}+{}^{\mathsf{t}}\overline{\mathsf{S}}\right)e^{\mathsf{S}u}\mathsf{Q}_j\right)\right)dV$$

and therefore,

$$\begin{aligned}\frac{d}{du}\left(\widetilde{KN_{k,\mathscr{F}}}\left(e^{\mathsf{S}u}\right)\right)_{u=0}&=\sum_j\varepsilon_j\int_M\mathtt{tr}({}^{\mathsf{t}}\overline{\mathsf{Q}_j}\mathsf{S}\mathsf{Q}_j),\\&=\sum_j\varepsilon_j\int_M\mathtt{tr}({}^{\mathsf{t}}\overline{\mathsf{Q}_j}\,{}^{\mathsf{t}}\mathsf{S}\mathsf{Q}_j)-\sum_j\varepsilon_j\frac{r-r_j}{N}\int_M\mathtt{tr}\left(\mathsf{S}\right),\\&=\langle\mu_{\mathscr{F},k}\left(\mathsf{Q}_0,\dots,\mathsf{Q}_{m-1}\right),\mathsf{S}\rangle.\end{aligned}$$

Finally, $\widetilde{KN_{k,\mathscr{F}}}(Id)=0$, which allows us to conclude. □

3.2 Balanced metrics for holomorphic filtrations

We are going to see that the two functionals $\widetilde{KN_{k,\mathscr{F}}}$ and $\widetilde{F_{k,\mathscr{F}}}$ are simultaneously proper. At this point, we will need the following classical result of potential theory,

Theorem 3.21 Set $Ka(M,\omega') = \{\varphi \in C^\infty(M,\mathbb{R}) : \omega' + i\partial\overline{\partial}\varphi > 0\}$. There exist some constants $\alpha_M, C(M,\omega,\omega') > 0$ such that for all $\varphi \in Ka(M,\omega')$, one has

$$\int_M e^{-\alpha_M(\varphi - \sup_M \varphi)} \frac{\omega^n}{n!} \leq C.$$

Lemma 5 *There exist some constants $(\gamma_i)_{i=1..3}$ such that for all $g \in SL(N)$*

$$\widetilde{F_{k,\mathscr{F}}}(g) - \gamma_1 \geq \frac{1}{\gamma_3}\widetilde{KN_{k,\mathscr{F}}}(g) \geq \widetilde{F_{k,\mathscr{F}}}(g) - \gamma_2.$$

Proof Let s_i be a basis of $H^0(\mathcal{F} \otimes L^k)$, at the point p,

$$\varphi_j(p) = \log \sum_{1 \leq i_1 < ... < i_{r-r_j} \leq N} \left\| \pi_j(g \cdot s_{i_1}(p)) \wedge ... \wedge \pi_j(g \cdot s_{i_{r-r_j}}(p)) \right\|^2.$$

Then φ_j belongs to the Kähler cone $Ka(M, c_1(\det(\mathcal{F} \otimes L^k/\mathcal{F}_j \otimes L^k)))$, and Theorem 3.21 asserts that there exist two real constants $\alpha_M > 0$ and $C > 1$ such that

$$\int_M e^{-\alpha_M(\varphi_j - \sup_M \varphi_j)} \frac{\omega^n}{n!} < C,$$

which implies that

$$\log\left(\int_M e^{-\alpha_M(\varphi_j - \sup_M \varphi_j)} \frac{\omega^n}{n!}\right) < C'.$$

Now by concavity of log,

$$\begin{aligned}\int_M \varphi_j dV &\geq \int_M \left(\sup_M \varphi_j\right) dV - \frac{1}{\beta(M,k,\omega)}, \\ &\geq V \log\left(\sup_{p\in M} \sum_{i_1 < ... < i_{r-r_j}} \left\| \pi_j(g s_{i_1}(p)) \wedge ... \wedge \pi_j(g s_{i_{r-r_j}}(p)) \right\|^2\right) \\ &\quad - \frac{1}{\beta(M,k,\omega)}.\end{aligned}$$

Indeed, by concavity of log, we have also

$$\log \int_M \sum_{1 \leq i_1 < ... < i_{r-r_j} \leq N} \left\| \pi_j(g \cdot s_{i_1}) \wedge ... \wedge \pi_j(g \cdot s_{i_{r-r_j}}) \right\|^2 dV \geq \int_M \varphi_j dV.$$

Now, summing previous inequalities for all j, we obtain the lemma with γ_i depending on the data $\{k, \mathscr{F}, L, dV\}$. $\square$

Definition 12 (Balanced metrics for holomorphic filtrations)
– Let $\mathbf{p} = (i_{k,0}, ..., i_{k,m-1}) \in \mathbf{\Pi}$ *be the point induced by the metric* $H \in Met(H^0(M, \mathcal{F} \otimes L^k))$*. If* $\mu_{\mathscr{F},k}(\mathbf{p}) = 0$ *then the holomorphic filtration* $\mathscr{F}$ *and* H *are said to be* k*-balanced.*
– If $H \in Met(H^0(M, \mathcal{F} \otimes L^k))$ *is* k*-balanced, the metric* $h \in Met(\mathcal{F} \otimes L^k)$ *given by* $h = FS_k(H)$ *is said to be* k*-balanced.*
We will say that the filtration $\mathscr{F}$ *is balanced if there exists an integer* k_0 *such that for all* $k \geq k_0$*,* $\mathscr{F}$ *is* k*-balanced.*

Since the sections $s_i \in H^0(M, \mathcal{F} \otimes L^k)$ are also coordinate sections of the universal bundle, we see that the metric $H \in Met(H^0(M, \mathcal{F} \otimes L^k))$ is balanced if and only if it is a fixed point of the map $Hilb_\omega \circ FS_k$. From a similar way, $h \in Met(\mathcal{F} \otimes L^k)$ is k-balanced if and only if it is a fixed point of the map $FS_k \circ Hilb_\omega$.

The balanced condition for a holomorphic filtration $\mathscr{F}$ can be translated in terms of the Bergman kernel of the bundle $\mathcal{F}$. We shall now make explicit what we mean by 'Bergman kernel' in the following definition.

Definition 13 *The Bergman kernel of a globally generated bundle* (F, h_F) *is an endomorphism of the bundle associated to the* L^2 *orthonormal projection from the space of sections* $L^2(M, F)$ *onto the space of holomorphic sections* $H^0(M, F)$*,*

$$\widehat{\mathsf{B}}_{F,h_F} = \sum_{i=1}^{h^0(M,F)} s_i \langle ., s_i \rangle_{h_F} \in C^\infty(M, End(F)) \tag{6}$$

where s_i *is any basis of* $H^0(M, F)$*, orthonormal for* $Hilb_\omega(h_F)$*.*

The following result will allow us to consider the balanced condition on the space of infinite dimension $Met(\mathcal{F})$ as the vanishing of a certain moment map related to the action of the Gauge group of the bundle $\mathcal{F}$. We will denote $\mathscr{F} \otimes L^k$ the associated holomorphic filtration obtained by tensorizing each subbundle of the filtration $\mathscr{F}$ by L^k.

Lemma 6 *The holomorphic filtration* $\mathscr{F}$ *of length* m *is balanced if and only if there exists an integer* $k_0 > 0$ *such that for all* $k \geq k_0$*, there exists a hermitian balanced metric* $h_k \in Met(\mathcal{F} \otimes L^k)$ *such that we have*

$$\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k, h_k} + \epsilon_k \sum_{j=1}^{m-1} \varepsilon_j \pi_{j,h_k}^{\mathscr{F}\otimes L^k} = \frac{N + \epsilon_k \sum_{j=1}^{m-1} \varepsilon_j r_j}{rV} Id_{\mathcal{F}\otimes L^k} \tag{7}$$

where $\epsilon_k = \frac{\chi(\mathcal{F}\otimes L^k)}{Vr - V\sum_{j=1}^{m-1} \varepsilon_j r_j}$.

Proof Assume that H is a balanced metric on $H^0(M, \mathcal{F} \otimes L^k)$ and s_i is an orthonormal basis of the space $H^0(M, \mathcal{F} \otimes L^k)$ for H. Let h be the quotient metric on $\mathcal{F} \otimes L^k$ induced by $V \twoheadrightarrow \mathcal{F} \otimes L^k$ and $FS_k(H)$ the metric on $\mathcal{F} \otimes L^k$ constructed as before. Remember at that point that the Bergman kernel is independent of the choice of an orthonormal

basis. We choose now a H-orthonormal basis of sections of $H^0(\mathcal{F}\otimes L^k)$ by the following procedure: let $s_1,...,s_{r_1}$ be orthogonal to the kernel of $(\pi_{1,h}^{\mathscr{F}\otimes L^k}\circ ev_p)$ with H-norm 1, and $s_{r_1+1},...,s_{r_2}\in\ker(\pi_{1,h}^{\mathscr{F}\otimes L^k}\circ ev_p)$ be orthogonal to the kernel of $(\pi_{2,h}^{\mathscr{F}\otimes L^k}\circ ev_p)$, and so on until the sections $s_{r_{m-1}+1},...,s_r\in\ker(\pi_{m-1,h}^{\mathscr{F}\otimes L^k}\circ ev_p)$ are orthogonal to the kernel of $(\pi_{m,h}^{\mathscr{F}\otimes L^k}\circ ev_p)$. Eventually, we can do the assumption that $s_i(p)=0$ for $i>r$. Introduce the sets $\mathscr{I}_j=\{r_{j-1}+1,...,r_j\}$ and the map $f:i\mapsto j$ where j is such that $i\in\mathscr{I}_j$. Hence, by (5), we compute

$$\begin{aligned}|s_i(p)|_h &= \frac{N}{Vr-V\sum_{j>0}\varepsilon_j r_j},\\ |s_i(p)|_{FS_k} &= \frac{N}{Vr-V\sum_{j>0}\varepsilon_j r_j}-\frac{N}{Vr-V\sum_{j>0}\varepsilon_j r_j}\varepsilon_{f(i)}.\end{aligned}$$

Note that the term $\frac{s_i\otimes s_i^{*FS_k}}{|s_i|^2_{FS_k}}\in End(\mathcal{F}\otimes L^k)$ is simply the orthogonal projection onto the image of s_i respectively to the metric FS_k (and also h). At $p\in M$,

$$\begin{aligned}\sum_{i=1}^N s_i\langle .,s_i\rangle_{FS_k} &= \sum_{i=1}^N s_i\langle .,s_i\rangle_h+\sum_i\Big(s_i\langle .,s_i\rangle_{FS_k}-s_i\langle .,s_i\rangle_h\Big),\\ &= Cst\times Id-\sum_i\left(\frac{s_i\otimes s_i^{*FS_k}}{|s_i|^2_{FS_k}}\frac{N}{Vr-V\sum_j\varepsilon_j r_j}\varepsilon_{f(i)}\right),\\ &= Cst\times Id-\frac{N}{Vr-V\sum_j\varepsilon_j r_j}\sum_j\varepsilon_j\pi_{j,h_k}^{\mathscr{F}\otimes L^k}. \end{aligned}\tag{8}$$

Here we have used the fact that for the quotient metric, the Bergman kernel is constant, since it can be considered as the identity isomorphism of the universal vector bundle over the Grassmannian. Clearly, this implies the existence of a metric $h_k=FS_k$ on the bundle $\mathcal{F}\otimes L^k$ satisfying (7).

Conversely, if (7) is satisfied, as the s_i are coordinate sections of the universal bundle, we obtain that they are also $Hilb_\omega(FS_k)$-orthonormal, i.e. they are orthonormal for

$$\int_M\left\langle\frac{N}{Vr-V\sum_j\varepsilon_j r_j}\left(Id-\sum\nolimits_j\varepsilon_j\pi_{j,h}^{\mathscr{F}\otimes L^k}\right).,.\right\rangle_h dV,$$

and therefore $Hilb_\omega(FS_k)$ is a metric on $H^0(M,\mathcal{F}\otimes L^k)$ which is a zero of the moment map $\mu_{\mathscr{F},k}$. □

Theorem 2 *Let $\mathscr{F}$ be a holomorphic filtration of length m over a projective manifold. Then $\mathscr{F}$ is $\mathbf{R}$-Gieseker stable if and only if $Aut(\mathscr{F})=\mathbb{C}$ and for k large, there exists a metric $h_k\in Met(\mathcal{F}\otimes L^k)$ such that*

$$\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k,h_k}+\epsilon_k\sum_{j=1}^{m-1}\varepsilon_j\pi_{j,h_k}^{\mathscr{F}\otimes L^k}=\frac{N+\epsilon_k\sum_{j=1}^{m-1}\varepsilon_j r_j}{rV}Id_{\mathcal{F}\otimes L^k}$$

where

$$\epsilon_k = \frac{\chi(\mathcal{F} \otimes L^k)}{Vr - V\sum_{j>0} \varepsilon_j r_j}.$$

Proof One already knows by Theorem 3.11 that the zeros of the moment map $\mu_{\mathscr{F},k}$ are the critical points of $\widetilde{KN_{k,\mathscr{F}}}$. To apply Kempf-Ness stability criterion [28], we simply need to remark that the functionals $\widetilde{KN_{k,\mathscr{F}}}$ and $\widetilde{F_{k,\mathscr{F}}}$ are simultaneously proper by Lemma 5. We are done with Lemma 6. □

4 $\boldsymbol{\tau}$-Hermite-Einstein metrics and holomorphic filtrations

We now consider the Hermite-Einstein equation for a holomorphic filtration $\mathscr{F}$ of length m,

$$\sqrt{-1}\Lambda F_h + \sum_{i=1}^{m-1} \tau_i \pi_{h,i}^{\mathscr{F}} = \mu_{\boldsymbol{\tau}}(\mathscr{F}) Id \tag{9}$$

on a smooth projective manifold M of complex dimension n. The goal of this part is to give an approximation of the metric solution of equation (9) using the balanced metrics that we have just defined. We will need the following expansion proved in [12, 39, 41] of the Bergman kernel of $\mathcal{F} \otimes L^k$ when $k \to \infty$,

Theorem 4.02 Let (M, ω) be a Kähler manifold and (L, h_L) an ample line bundle on M such that ω represents the curvature of L. Let $(\mathcal{F}, h_{\mathcal{F}})$ be a holomorphic hermitian vector bundle. For any integer $\alpha \geq 0$, we have the following asymptotic expansion when $k \to +\infty$ of the Bergman kernel $\widehat{\mathsf{B}}_{h_{\mathcal{F}} \otimes h_{L^k}}$

$$\left\| \widehat{\mathsf{B}}_{h_{\mathcal{F}} \otimes h_{L^k}} - k^n Id_{r\times r} - \left(\frac{1}{2} Scal(g) Id_{r \times r} + \sqrt{-1}\Lambda F_{h_{\mathcal{F}}}\right) k^{n-1} \right\|_{C^\alpha} \leq C_\alpha k^{n-2} \tag{10}$$

where $Scal\,(g)$ denotes the scalar curvature of the metric g associated to the Kähler form

$$\omega = \frac{i}{2} \sum g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.$$

This estimate is uniform on M when $h_{\mathcal{F}}$ and h_L belong to a compact for the C^α topology.

Moreover, we denote in all the following for any integer k,

$$\epsilon_k = \frac{\chi(\mathcal{F} \otimes L^k)}{Vr - V\sum_{j=1}^{m-1} \varepsilon_j r_j}.$$

4.1 Action of the Gauge group

Let $\mathcal{A}(\mathcal{F})$ be the space of C^∞ connections on the vector bundle $\mathcal{F}$. By the Newlander-Nirenberg's theorem, it is equivalent to considering a holomorphic structure on the $\mathbb{C}^r$ vector bundle $\mathcal{F}$ or an operator $\overline{\partial}$

$$\overline{\partial} : \Omega^0(\mathcal{F}) \to \Omega^{0,1}(\mathcal{F})$$

with $\overline{\partial}(f.s) = \overline{\partial} f.s + f.\overline{\partial} s$ and $\overline{\partial}^2 = 0$. We will denote $\mathcal{A}(\mathcal{F}, h_{\mathcal{F}})$ the space of smooth connections which are compatible with the hermitian metric $h_{\mathcal{F}}$ (i.e. unitary). Any connection on the holomorphic vector bundle $\mathcal{F}$ which is integrable, i.e. belongs to the subset

$$\mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}}) = \{A \in \mathcal{A}(\mathcal{F}, h_{\mathcal{F}}) : F_A^{0,2} = F_A^{2,0} = 0\}$$

where $F_A \in \Omega^2(M, End(\mathcal{F}))$ denotes the curvature of the connection, defines a holomorphic structure on $\mathcal{F}$. It is well known that a holomorphic vector bundle equipped with a hermitian metric admits a unique connection which is compatible with the holomorphic structure (i.e. integrable) and the hermitian structure $h_{\mathcal{F}}$, called Chern connection (see [30] for the details). This means that there exists an isomorphism (see [4, Section 8])

$$A^{1,1}(\mathcal{F}, h_{\mathcal{F}}) \tilde{\to} \mathcal{C}(\mathcal{F}),$$

where we have set

$$\begin{aligned}\mathcal{C}(\mathcal{F}) = \{\nabla^{0,1} : \Omega^0(\mathcal{F}) \to \Omega^{0,1}(\mathcal{F}) \text{ s.t. } &\left(\nabla^{0,1}\right)^2 = 0,\\ &\nabla^{0,1}(f.s) = \nabla^{0,1} f.s + f.\nabla^{0,1} s\}.\end{aligned}$$

$\mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}})$ is a subvariety (possibly with singularities) of infinite dimension of the symplectic space $\mathcal{A}(\mathcal{F})$ which can be equipped of the symplectic structure (cf. [4, p.587] or [15]):

$$\Omega(A, B) = \int_M Tr(A \wedge B) \frac{\omega^{n-1}}{(n-1)!}$$

for any $A, B \in \Omega^1(End(\mathcal{F}))$. One denotes by $\mathcal{G}$ the Gauge group of $\mathcal{F}$, i.e. the group of unitary automorphisms of $\mathcal{F}$:

$$\mathcal{G} = \{U \in C^\infty(GL(\mathcal{F})) : U^* U = I\}.$$

The complexified Gauge group, i.e. the space of smooth sections of automorphisms $\mathcal{G}^{\mathbb{C}} = C^\infty(GL(\mathcal{F}))$ acts (on the right) on $\mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}})$ by

$$\begin{aligned}\overline{\partial}_{g(A)} &= g^{-1} \circ \overline{\partial}_A \circ g, \\ \partial_{g(A)} &= {}^{\mathsf{t}}\overline{g} \circ \partial_A \circ \left({}^{\mathsf{t}}\overline{g}\right)^{-1},\end{aligned} \tag{11}$$

where $A \in \mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}})$, $g \in \mathcal{G}^{\mathbb{C}}$ and ∂_A is the $(1,0)$ part of the covariant derivate induced naturally from A. In particular, the action of $\mathcal{G}^{\mathbb{C}}$ is holomorphic on $\mathcal{C}(\mathcal{F})$ which admits a complex structure. Therefore $\mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}})$ inherits a complex structure for which Ω is a Kähler metric.

4.2 Limit of a sequence of balanced metrics

We may naturally ask what can be the limit of a sequence of balanced metrics. In this section, we show that if this limit does exist, it is necessarily a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric.

Definition 14 *Let (Ξ, ω) be a complex manifold and let $\mathscr{F}$ be a holomorphic filtration of length m over Ξ. A hermitian metric h on $\mathcal{F}$ is said to be conformally $\boldsymbol{\tau}$-Hermite-Einstein if the curvature F_h of the Chern connection associated to $h \in Met(\mathcal{F})$ satisfies the equation*

$$\sqrt{-1}\Lambda_\omega F_h + \sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h,i} = \lambda_h Id_{\mathcal{F}}, \tag{12}$$

where λ_h is a real valued function.

Proposition 3 *Assume that (Ξ, ω) is a compact Kähler manifold. If h is conformally $\boldsymbol{\tau}$-Hermite-Einstein then there exists $f \in C^\infty(M, \mathbb{R})$ unique up to a constant, such that the new metric $e^f \cdot h$ is $\boldsymbol{\tau}$-Hermite-Einstein with parameter $\lambda = \frac{1}{V}\int_M \lambda_h dV$ in equation (12).*

Proof Indeed, with this conformal change, we get

$$\sqrt{-1}\Lambda_\omega F_{e^f \cdot h} = \sqrt{-1}\Lambda_\omega F_h + \sqrt{-1}\Lambda_\omega \partial\overline{\partial}(f) Id$$

and

$$\pi^{\mathscr{F}}_{e^f \cdot h, i} = \pi^{\mathscr{F}}_{h,i}.$$

Now, it can be found a function $f \in C^\infty(M, \mathbb{R})$ by classical theory of elliptic operators over a compact manifold M (see [30, Corollary 7.2.9]) such that

$$\sqrt{-1}\Lambda_\omega \partial\overline{\partial}(f) = \lambda_h - \frac{1}{V}\int_M \lambda_h dV.$$

□

Notation To a sequence of balanced metrics $h_k \in Met(\mathcal{F} \otimes L^k)$ for a holomorphic filtration $\mathscr{F}$, we shall associate the sequence of normalized metrics $\boldsymbol{h}_k = h_k \otimes h_L^{-k} \in Met(\mathcal{F})$ that will still be qualified as balanced metrics.

We use use the asymptotic expansion provided by Theorem 4.02 and the previous proposition to get

Theorem 3 *Let $\mathscr{F}$ be a balanced holomorphic filtration of length m over M. If the sequence of balanced metrics $\boldsymbol{h}_k \in Met(\mathcal{F})$ admits a limit h_∞ in C^2 topology when $k \to \infty$, then the metric h_∞ is conformally $\boldsymbol{\tau}$-Hermite-Einstein satisfying*

$$\sqrt{-1}\Lambda F_{h_\infty} + \sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h,i} = \left(\mu_{\boldsymbol{\tau}}(\mathcal{F}) + \frac{1}{2}\left(\int_M c_1(M)\,\omega^{n-1} - Scal(g)\right)\right) Id_{\mathcal{F}}, \tag{13}$$

and up to a conformal change, this metric is $\boldsymbol{\tau}$-Hermite-Einstein.

4.3 Stable holomorphic filtrations and natural moment maps

We adapt the method of [16] to make apparent the balanced condition as the vanishing of two moment maps, one induced by the unitary group and the other one by the Gauge group of the vector bundle $\mathcal{F}$. We know that the space $C^\infty(M,\mathcal{F})$ of smooth sections of $\mathcal{F}$ has a natural symplectic form $\Omega_{[0]}$ associated to the hermitian metric $h_\mathcal{F}$ on $\mathcal{F}$:

$$\Omega_{[0]}(s_1,s_2) = 2\mathrm{Im}\left(\int_M \langle s_1, s_2\rangle_{h_\mathcal{F}} dV\right).$$

It is not difficult to check that

$$\mu_{C^\infty(M,\mathcal{F})}(s) = \sqrt{-1}s\langle ., s\rangle_{h_\mathcal{F}}\frac{\omega^n}{n!} \in \Omega^{2n}(M, End(\mathcal{F})) \simeq Lie(\mathcal{G}^\mathbb{C})^*$$

is a moment map associated to the action of the group $\mathcal{G}$ on $C^\infty(M,\mathcal{F})$.

For a holomorphic filtration $\mathscr{F}$, we can consider a family $\theta_i : \mathcal{F}_i \to \mathcal{F}$ of smooth sections of the bundle in Grasmanians that we denote $Gr(r_i,\mathcal{F})$ and whose fibers in $p \in M$ are the r_i planes of $\mathcal{F}_{|p}$. Such a section θ_i gives naturally a projection $h_\mathcal{F}$-orthogonal onto the orthogonal of its kernel with respect to $h_\mathcal{F}$, i.e. the projection $\pi^{\mathscr{F}}_{h_\mathcal{F},i}$. Moreover, the metric $h_\mathcal{F}$ on the fiber $\mathcal{F}_{|p}$ induces a Kähler form $\omega^{Gr}_{h_\mathcal{F}}$ on $Gr(r_i,\mathcal{F}_{|p})$. We obtain a symplectic form using the evaluation map $ev_i : C^\infty(M, Gr(r_i,\mathcal{F}))\times M \to Gr(r_i, r)$ and the projection on the first component $p_1 : C^\infty(M, Gr(r_i,\mathcal{F}))\times M \to C^\infty(M, Gr(r_i,\mathcal{F}))$,

$$\Omega_{(i)} = (p_1)_*\left(ev_i^*(\omega^{Gr}_{h_\mathcal{F}})\wedge dV\right).$$

The action of the Gauge group on $C^\infty(M, Gr(r_i,\mathcal{F}))$ respectively to $\Omega_{(i)}$ is then given by

$$\mu_{C^\infty(M,Gr(r_i,\mathcal{F}))}(\theta_i) = \sqrt{-1}\pi^{\mathscr{F}}_{h_\mathcal{F},i}\frac{\omega^n}{n!} \in Lie(\mathcal{G}^\mathbb{C})^*.$$

We know that there exists for k large enough an embedding i_k of M into the Grassmannian $Gr(N,r)$ using the holomorphic sections of $\mathcal{F}\otimes L^k$. Nevertheless the action of $\mathcal{G}$ on $C^\infty(M,\mathcal{F}\otimes L^k)$ does not preserve the set of holomorphic sections for a connection $A \in \mathcal{A}^{1,1}(\mathcal{F}\otimes L^k, h_\mathcal{F}\otimes h_L^k)$ defined *a priori*. Note that in general, the dimension of the space of holomorphic sections of of $\mathcal{F}\otimes L^k$ depends on the choice of the connection A. In order to consider global holomorphic sections and their variations with respect to the Gauge group, we are constrained to modify simultaneously the considered connection. But for any connection A and for k large enough, there exists an open set of the complex orbit of A in $\mathcal{G}^\mathbb{C}$ such that for any connection belonging to this set, $\dim(H^i(M,\mathcal{F}\otimes L^k)) = 0$ (by semi-continuity [37, Section 9.3]) and $\dim(H^0(M,\mathcal{F}\otimes L^k))$ is constant. Finally for such a k, we need to introduce the following manifold of infinite dimension,

Definition 15 *Let $\mathscr{F}$ be a holomorphic filtration of length m, and $\mathcal{Q}_0$ the subset of*

$$C^{\infty}(M, \mathcal{F} \otimes L^k)^N \times \mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}}) \times \prod_{i=1}^{m-1} C^{\infty}(M, Gr(r_i, \mathcal{F}))$$

formed by $(N+m)$-tuples of the form

$$\Big\{s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}\Big\}$$

such that the sections $(s_i)_{i=1..N}$ are linearly independent, and

$$\overline{\partial}_A s_i = 0 \quad \forall i = 1, .., n \tag{14}$$
$$\overline{\partial}_{[A]} \theta_j = 0 \quad \forall j = 1, .., m-1, \tag{15}$$

where $\overline{\partial}_A$ represents the $(0,1)$ part of the covariant derivative induced naturally using the unitary connection A and the Chern connection on L over the space $C^{\infty}(M, \mathcal{F} \otimes L^k)$ and $\overline{\partial}_{[A]}$ represents the $(0,1)$ part of the covariant derivative induced naturally using A on $C^{\infty}(M, Gr(r_i, \mathcal{F}))$.

Under these conditions, the diagonal action of $\mathcal{G}$ preserves $\mathcal{Q}_0$. Also notice that the unitary group $U(N)$ acts naturally on $\mathcal{Q}_0$ by

$$(\mathsf{u}_{ij}) * \{s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}\} = \left\{\sum_{j=1}^{N} \mathsf{u}_{1j} s_j, ..., \sum_{j=1}^{N} \mathsf{u}_{Nj} s_j, A, \theta_1, ..., \theta_{m-1}\right\},$$

and if we denote $h = h_{\mathcal{F}} \otimes h_{L^k}$ then the moment map for this action is,

$$\mu_{U(N)}(s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) = \int_M \langle s_i, s_j \rangle_h dV.$$

Let $\pi_0 : \mathcal{Q}_0 \to C^{\infty}(M, \mathcal{F} \otimes L^k)^N \times \prod_{i=1}^{m-1} C^{\infty}(M, Gr(r_i, \mathcal{F}))$ be the natural projection. Actually, π_0 is an immersion. Indeed, one has by (14) that $0 = \overline{\partial}_A(\varepsilon s_i) + \varepsilon\overline{\partial}_A(s_i) = \varepsilon\overline{\partial}_A(s_i)$ (and similary $\varepsilon\overline{\partial}_A(\theta_i) = 0$) for a small variation of $\big(s_1, .., s_N, \overline{\partial}_A, \theta_1, .., \theta_{m-1}\big)$. Since i_k is an embedding, we get that $\varepsilon\overline{\partial}_A = 0$ and hence $d\pi_0$ is injective, which gives the immersion property.

We consider the symplectic standard form $\Omega_{[k]}$ on $C^{\infty}(M, \mathcal{F} \otimes L^k)$. We take the sum of $\Omega_{[k]}$ over N copies of the spaces $C^{\infty}(M, \mathcal{F} \otimes L^k)$ and consider the symplectic form $\Omega_{(i)}$ on each $C^{\infty}(M, Gr(r_i, \mathcal{F}))$ with weight $\epsilon_k \varepsilon_i$. We get a symplectic form that we can lift using the injective immersion π_0. We denote $\Omega_{\mathcal{Q}_0}$ the symplectic form obtained on $\mathcal{Q}_0$. The action of $\mathcal{G}$ over $\mathcal{Q}_0$ admits a moment map associated to $\Omega_{\mathcal{Q}_0}$,

$$\mu_{\mathcal{G}}(s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) = \sum_{i=1}^{N} s_i \langle ., s_i \rangle_h + \epsilon_k \sum_{i=1}^{m-1} \varepsilon_i \pi_{h,i}^{\mathscr{F} \otimes L^k}.$$

Indeed, $\mathcal{Q}_0$ admits a Kähler structure by [24, Theorem 3] and with the fact $\mathcal{A}^{1,1}(\mathcal{F}, h_{\mathcal{F}})$ admits a complex structure. Moreover, the actions of $\mathcal{G}$ and $U(N)$ commute and these two groups have center of dimension 1, given respectively by the constant functions and the multiples of identity. This allows us to restrict to the action $SU(N)$ by considering another natural moment map with values in the Lie algebra $\sqrt{-1}\mathfrak{su}(N)$,

$$\mu_{SU(N)}(s_1, .., s_N, A, \theta_1, .., \theta_{m-1}) = \int_M \langle s_i, s_j\rangle_h dV - \frac{1}{N}\left(\sum_i \int_M |s_i|_h^2 dV\right)\delta_{ij}.$$

4.4 Complex orbits and double symplectic quotient

The moment map for the action of the product $\mathcal{G} \times SU(N)$ on $\mathcal{Q}_0$ will be given by the direct sum $\mu_{\mathcal{G}} \oplus \mu_{SU(N)}$. For a real number λ, we will consider the symplectic quotient

$$\mathcal{Q}_0//(\mathcal{G} \times SU(N)) = \frac{\mu_{\mathcal{G}}^{-1}(\lambda Id) \cap \mu_{SU(N)}^{-1}(0)}{\mathcal{G} \times SU(N)}.$$

This must be understood as the symplectic quotient of $\mathcal{Q}_0$ by $\mathcal{G}$ in a first step, via

$$\mathcal{Q}_0//\mathcal{G} = \mu_{\mathcal{G}}^{-1}(\lambda Id)/\mathcal{G}$$

which admits as we have seen, a natural symplectic structure. Secondly, we do the symplectic quotient of $\mathcal{Q}_0//\mathcal{G}$ by $SU(N)$, since $SU(N)$ acts naturally on $\mathcal{Q}_0//\mathcal{G}$. Every $\mathcal{G}^{\mathbb{C}}$-orbit in $\mathcal{Q}_0$ contains a point in $\mu_{\mathcal{G}}^{-1}(\lambda Id)$, (resp. $\mu_{SU(N)}^{-1}(0)$) unique up to the action of $\mathcal{G}$ (resp. $SU(N)$). Our situation is summed up by the following proposition.

Proposition 4 *There exists a metric k-balanced for $\mathscr{F}$ if and only if the complex orbit in $\mathcal{Q}_0$ given by the action of $\mathcal{G}^{\mathbb{C}} \times SL(N)$, contains a point in $\mu_{\mathcal{G}}^{-1}(\lambda Id) \cap \mu_{SU(N)}^{-1}(0)$ for all $\lambda > 0$. This is equivalent to saying that the complex orbit is represented by a point in the double symplectic quotient $\mathcal{Q}_0//(\mathcal{G} \times SU(N))$.*

Proof A point $z_0 = \{s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}\} \in \mathcal{Q}_0$ belongs to $\mu_{\mathcal{G}}^{-1}(\lambda Id)$ if and only if $\sum_i s_i\langle ., s_i\rangle_h + \epsilon_k \sum_i \varepsilon_i \pi_{h,i}^{\mathscr{F}} = \lambda Id$ and z_0 belongs to $\mu_{SU(N)}^{-1}(0)$ if and only if there exits a constant c such that the sections s_i form a L^2-orthonormal basis. Therefore we get $\lambda = \frac{1}{rV}\left(\chi(\mathcal{F} \otimes L^k) + \epsilon_k \sum_j \varepsilon_j r_j\right)$ by taking the trace and hence we obtain the balance condition of Lemma 6.

Consider the action $\cdot$ of $SU(N)$ on the symplectic quotient

$$\mathcal{Z} = \mathcal{Q}_0//\mathcal{G}.$$

Since $\mathcal{Q}_0$ is Kähler, $\mathcal{Z}$ admits a Kähler structure (in fact it is an orbifold if the stabilizers group is finite everywhere) since the extension of the action of $\mathcal{G}$ to $\mathcal{G}^{\mathbb{C}}$ preserves the

complex structure. For all point $z \in \mathcal{Z}$, the infinitesimal action of $\mathfrak{su}(N)$, provides us with an application

$$\nu_z^{\mathcal{Z},SU(N)} : \mathfrak{su}(N) \to T\mathcal{Z}_z.$$

Let us consider the following operator on $\mathfrak{su}(N)$,

$$\mathfrak{q}_z^{SU(N)} = \left(\nu_z^{\mathcal{Z},SU(N)}\right)^* \nu_z^{\mathcal{Z},SU(N)},$$

where $\left(\nu_z^{\mathcal{Z},SU(N)}\right)^*$ is the adjoint of $\nu_z^{\mathcal{Z},SU(N)}$ formed using the invariant metric on $\mathfrak{su}(N)$ and the metric on $T\mathcal{Z}_z$. Assume that the stabilizers of a point in $\mathcal{Z}$ under the action of $SU(N)$ are discrete; then $\nu_z^{\mathcal{Z},SU(N)}$ is injective and $\mathfrak{q}_z^{SU(N)}$ is invertible.

Notation Let Q be a hermitian matrix. The Hilbert-Schmidt norm and the operator norm for Q are given by

$$||\mathsf{Q}||^2 = \sum_{i,j} |\mathsf{Q}_{ij}|^2,$$

$$|||\mathsf{Q}||| = \sup_{||v|| \leq 1} \frac{|\mathsf{Q}v|}{|v|}.$$

Notation Let Λ_z (resp. $\boldsymbol{\Lambda}_z$) be the Hilbert-Schmidt norm (resp. operator norm) of $\left(\mathfrak{q}_z^{SU(N)}\right)^{-1} : \mathfrak{su}(N) \to \mathfrak{su}(N)$ with respect to the invariant euclidian metric on $\mathfrak{su}(N)$.

In particular, the inequality $\boldsymbol{\Lambda}_z \leq \lambda$ is induced from the fact that for all $\mathsf{A} \in \sqrt{-1}\mathfrak{su}(N)$ one has $|\mathsf{A}|^2 \leq \lambda \left|\nu_z^{\mathcal{Z},SU(N)}(\sqrt{-1}\mathsf{A})\right|^2_{T\mathcal{Z}}$.

We will need the following key result inspired directly from the formalism for moment maps from [16].

Proposition 5 *Let $z_0 \in \mathcal{Z}$ and $\lambda, \delta > 0$ be two real numbers such that*
1. $\lambda||\mu_{SU(N)}(z_0)|| < \delta$;
2. $\boldsymbol{\Lambda}_z \leq \lambda$ pour tout $z = e^{i\mathsf{S}} \cdot z_0$ and $||\mathsf{S}|| \leq \delta$.
Then, there exists a zero $z_1 = e^{i\mathsf{S}'} \cdot z_0$ of $\mu_{SU(N)}$,

$$\mu_{SU(N)}(z_1) = 0$$

with $||\mathsf{S}'|| \leq \lambda||\mu(z_0)||$. Here $||.||$ is the norm induced by the standard $SU(N)$-invariant inner product on $\mathfrak{su}(N)$.

Proof See [16, Proposition 17]. Indeed, let z_0 be a point in $\mathcal{Z}$ and denote $(z_t)_{t>0}$ the trajectory of z_0 along the flow $-\overrightarrow{Grad}(||\mu_{SU(N)}(z)||^2)$ and set $\mathcal{Z}^{min} = \{z_0 \in \mathcal{Z} : \lim_{t\to+\infty} z_t \in \mu^{-1}(0)\}$. Then it is well-known (see [29, Theorem 7.4]) that the complex orbits of the zeros of $\mu_{SU(N)}$ can be identified with the set $\mathcal{Z}^{min}$. Finally, this proposition is an effective version of this result. □

The aim of the next three sections is the approximation Theorem 5 of a metric solution of the $\boldsymbol{\tau}$-Hermite-Einstein equation using the tools defined in sections 4.1, 4.2, 4.3, 4.4. We are now ready to give the main ideas to prove this result.

An irreducible $\boldsymbol{\tau}$-Hermite-Einstein holomorphic filtration is in particular Gieseker $\mathbf{R}$-stable when one has fixed $R_i = \tau_i k^{n-1}$. Now, we are seeking to construct for such a filtration a sequence of balanced metrics which converge towards the (conformally) τ-Hermite-Einstein metric that satisfies equation (13). To a k-balanced filtration $\mathscr{F}$, we know that there corresponds a point in a certain space of parameters, inside a complex orbit (under the action of $\mathcal{G}^{\mathbb{C}} \times SL(N)$) and this point is a zero of the moment map $\mu_{\mathcal{G}} \oplus \mu_{SU(N)}$ as previously defined. Finally this leads us to look for this point.

- From one side, via Propositions 6 and 7, we get a point in the symplectic quotient by $\mathcal{G}$, i.e. a zero of $\mu_{\mathcal{G}}$: the existence *a priori* of a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric will permit us to construct an 'almost' balanced metric $h_{k,q}$ that will provide us with a point in $\mathcal{Z}$. To such a point will correspond a metric denoted $\widetilde{h_q}$.
- From another side, we will switch in a finite dimensional problem by looking for a zero (unique up to action of $SU(N)$) of the moment map $\mu_{SU(N)}$ in a $SL(N)$-orbit. Thus, we will study the gradient flow of $||\mu_{SU(N)}||^2$ on $\mathcal{Z}$ and give an estimate of $\boldsymbol{\Lambda}_z$ in order to apply Proposition 5.

Therefore, we finally obtain a k-balanced metric and also by construction, the convergence of this sequence of metrics (when $k \to \infty$) towards the conformally $\boldsymbol{\tau}$-Hermite-Einstein metric solution of (13).

4.5 Construction of almost balanced metrics

From now on and until the end of this section, we will consider that there exists a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric h_∞ for the irreducible filtration $\mathcal{F}$. By singular perturbation of the conformally $\boldsymbol{\tau}$-Hermite-Einstein metric, we can get a metric almost balanced, as will be made explicit in the following proposition.

Proposition 6 *Let $\mathscr{F}$ be an irreducible holomorphic filtration over M of length m such there exists a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric h_∞ on $\mathcal{F}$ satisfying equation (13). Then, there exists a family of smooth hermitian endomorphisms $(\boldsymbol{\eta}_i)_{i\in\mathbb{N}} \in C^\infty(End(\mathcal{F}))$ such that the metrics defined on $\mathcal{F}$ for all $q \geq 1$ by*

$$\langle .,. \rangle_{h_{k,q}} = \left\langle \left(Id + \sum_{i=1}^{q} \frac{1}{k^i}\boldsymbol{\eta}_i \right) .,. \right\rangle_{h_\infty} \otimes h_{L^k},$$

are hermitian and smooth for k large enough and there exists a constant $C_{q,\alpha}$ such that

$$\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k, h_{k,q}} + \epsilon_k \sum_{i=1}^{m-1} \varepsilon_i \pi^{\mathscr{F}\otimes L^k}_{h_{k,q},i} = \frac{\chi(\mathcal{F}\otimes L^k) + \epsilon_k \sum_i \varepsilon_i r_i}{rV} Id + \boldsymbol{\sigma}_q(k), \qquad (16)$$

where $\|\boldsymbol{\sigma}_q(k)\|_{C^{\alpha+2}} \leq C_{q,\alpha} k^{n-q-1}$. *The metrics* $h_{k,q}$ *will be called 'almost balanced'. Here* $C_{q,\alpha}$ *is a constant that depends only on* q, α, h_∞ *and* ω.

Proof First of all, we know that under these assumptions, $\mathscr{F}$ is simple by Lemma 1. The Theorem 4.02 asserts that we have an asymptotic expansion in the variable k of the Bergman kernel

$$\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k, h_\infty \otimes h_{L^k}} = k^n Id + \mathsf{a}_1(h_\infty) k^{n-1} + ... + \mathsf{a}_q(h_\infty) k^{n-q} + \mathbf{O}(k^{n-q-1}),$$

and the a_i are polynomial expressions of the curvature tensors of h_∞ and h_L and their covariant derivatives. The approximation term is uniformly bounded in C^α norm when $h_\infty \otimes h_L^k$ belongs to a bounded family in $C^{\alpha'}$ norm (where α' depends on α). We notice that we have also $\mathsf{a}_1(h_\infty) = \sqrt{-1}\Lambda_\omega F_{h_\infty}$.
Consequently, $\mathsf{a}_i\left(h_\infty\left(1+\boldsymbol{\eta}\right)\right) = \mathsf{a}_i(h_\infty) + \sum_{l=1}^q \mathsf{a}_{i,l}(\boldsymbol{\eta}) + O(\|\boldsymbol{\eta}\|_{C^s}^{q+1})$ with s sufficiently large depending on α and q. For all $(\boldsymbol{\eta}_i)_{i\in\mathbb{N}} \in C^\infty(End(\mathcal{F}))$, we can write

$$\mathsf{a}_i\left(h_\infty\left(1+\textstyle\sum_{j=1}^q \boldsymbol{\eta}_j k^{-j}\right)\right) = \mathsf{a}_i(h_\infty) + \sum_{l=1}^q \mathsf{b}_{i,l} k^{-l} + \mathbf{O}(k^{-q-1}),$$

where the $\mathsf{b}_{i,l}$ are multilinear expressions in $\boldsymbol{\eta}_j$ and their covariant derivatives, beginning by

$$\mathsf{b}_{i,1} = \mathsf{a}_{i,1}\left(\boldsymbol{\eta}\right).$$

If we now set $\mathsf{a}_i = \mathsf{a}_i(h_\infty)$, then we get

$$\begin{aligned}\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k, h_{k,q}} &= \sum_{p=0}^q k^{n-p}\mathsf{a}_p + \sum_{i,l=1}^r \mathsf{b}_{i,l} k^{n-i-l} + \mathbf{O}(k^{n-q-1}),\\ &= k^n + \mathsf{a}_1 k^{n-1} + (\mathsf{a}_2 + \mathsf{b}_{1,1})\, k^{n-2}\\ &\quad + k^{n-3}(\mathsf{a}_3 + \mathsf{b}_{1,2} + \mathsf{b}_{2,1}) + ... + \mathbf{O}(k^{n-q-1}),\end{aligned} \tag{17}$$

and we choose inductively the $\boldsymbol{\eta}_j$ in such a way that the coefficients that appear with k^{n-j} $(j < q)$ are constants, which means that the RHS of (17) is exactly (up to order k^{n-q-1})

$$\frac{\chi(\mathcal{F}\otimes L^k) + \epsilon_k \sum_i \varepsilon_i r_i}{rV} Id - \epsilon_k \sum_i \varepsilon_i \pi_{h_{k,q},i}^{\mathscr{F}\otimes L^k}.$$

We set the asymptotic expansions (in the variable k),

$$\begin{aligned}\frac{\chi(\mathcal{F}\otimes L^k) + \epsilon_k \sum_i \varepsilon_i r_i}{rV} Id &= \mathsf{c}_0 k^n + \mathsf{c}_1 k^{n-1} + ...\\ \epsilon_k \frac{1}{k} &= d_1 k^{n-1} + d_2 k^{n-2} + ...\end{aligned}$$

By Lemma 16, we get also the expansion

$$\begin{aligned}\sum_j \tau_j \pi^{\mathcal{F}\otimes L^k}_{h_{k,q},j} &= \sum_j \tau_j \pi^{\mathcal{F}}_{h_\infty,j} + \frac{1}{k} \Pi^{\mathscr{F},\boldsymbol{\tau}}_{h_\infty}(\boldsymbol{\eta}_1) + ... \\ &= \mathsf{e}_1 + k^{-1}\mathsf{e}_2 + ...\end{aligned}$$

where we have done the substitution $\epsilon_k \sum_j \varepsilon_j \pi^{\mathcal{F}\otimes L^k}_{h_{k,q},j} = \frac{\epsilon_k}{k} \sum_j \tau_j \pi^{\mathcal{F}\otimes L^k}_{h_{k,q},j}$. In particular, we have $d_1 = 1$ by our choice of ϵ_k.
From another point of view, we know that if F_H denotes the curvature of the H, then $F_{H(1+\varepsilon)} = F_H + \overline{\partial}\partial\varepsilon + O\left(||\varepsilon||^2\right)$, and therefore

$$\mathsf{b}_{1,1} = \sqrt{-1}\Lambda\left(\overline{\partial}\partial\boldsymbol{\eta}_1\right).$$

Moreover, when k is sufficiently large

$$\begin{aligned}e_1 &= \sum_j \tau_j \pi^{\mathcal{F}}_{h_\infty,j}, \\ e_2 &= \sum_j \tau_j \pi^{\mathcal{F}}_{h_\infty,j}\left(\boldsymbol{\eta}_1\right)\left(Id - \pi^{\mathcal{F}}_{h_\infty,j}\right).\end{aligned}$$

In order to get $\boldsymbol{\eta}_1$, we aim to solve

$$\mathsf{b}_{1,1} + d_1\mathsf{e}_2 = \mathsf{c}_2 - \mathsf{a}_2 - d_2\mathsf{e}_1.$$

But the operator $Q : u \mapsto \sqrt{-1}\Lambda\overline{\partial}\partial u + d_1 \Pi^{\mathscr{F},\boldsymbol{\tau}}_h(u)$ is elliptic of order 2. We can apply Lemma 18 noticing that $\int_M \mathrm{tr}(\mathsf{c}_2 - \mathsf{a}_2 - d_2\mathsf{e}_1) = 0$ and the endomorphism $(\mathsf{c}_2 - \mathsf{a}_2 - d_2\mathsf{e}_1)$ preserves the filtration. Then, we get a solution $\boldsymbol{\eta}_1$ which is self-adjoint since the term $\mathsf{c}_2 - \mathsf{a}_2 - d_2\mathsf{e}_1$ is also self-adjoint. Now, if we are looking for an almost balanced metric up to order 3, it is sufficient to solve

$$\mathsf{b}_{2,1} + d_1\mathsf{e}_3 = \mathsf{c}_3 - \mathsf{a}_3 - \mathsf{b}_{1,2} - d_3\mathsf{e}_1 - d_2\mathsf{e}_2.$$

We find all the $\boldsymbol{\eta}_j$ by solving at each step a differential equation of the form

$$\sqrt{-1}\Lambda\left(\overline{\partial}\partial\boldsymbol{\eta}_j\right) + \sum_j \tau_j \pi^{\mathcal{F}}_{h_\infty,j}\boldsymbol{\eta}_j(Id - \pi^{\mathcal{F}}_{h_\infty,j}) = \mathsf{c}_j - \mathsf{a}_j - P_j(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, ..., \boldsymbol{\eta}_{j-1}),$$

where P_j is self-adjoint, $\int_M \mathtt{tr}(P_j + \mathsf{a}_j - \mathsf{c}_j) = 0$, $(P_j + \mathsf{a}_j - \mathsf{c}_j)$ preserves the filtration, and P_j is totally determined by the $\boldsymbol{\eta}_l$ computed previously for $l < j$. The fact that $h_{k,q}$ is hermitian is clear since the endomorphisms a_i, the generalized Bergman kernel and the operator P_j are hermitian. □

In order to get from the conformally $\boldsymbol{\tau}$-Hermite-Einstein metric a point in the symplectic quotient $\mathcal{Z}$, we will need the following lemma.

Lemma 7 *Let*

$$\mathfrak{q} = (s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Q}_0$$

be a point corresponding to the holomorphic filtration $\mathscr{F}$*. Fix* $\widetilde{h}$ *a hermitian metric on* $\mathcal{F}\otimes L^k$ *and consider the map in the Sobolev space* $End(\mathcal{F})^{l,\alpha}$ *of hermitian endomorphisms of* $C^{l,\alpha}$ *class of* $\mathcal{F}$ *over the compact manifold* M*,*

$$\mathcal{B}_{\mathfrak{q},\widetilde{h},\rho} : \eta \mapsto \left(\sum_{i=1}^{N} s_i\langle ., s_i\rangle_{\widetilde{h}}\right)\eta + \rho\sum_{j=1}^{m-1}\frac{\tau_j}{k}\pi^{\mathscr{F}\otimes L^k}_{\widetilde{h}(\eta .,),j}$$

where $\rho \in \mathbb{R}$*. If we assume that*
– $\mathrm{tr}\left(\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}\right) = ck^n + O(k^{n-1})$*,*
– $0 \leq \rho_0 \leq \epsilon_k = \frac{\chi(\mathcal{F}\otimes L^k)}{Vr - V\sum_{j>0}\varepsilon_j r_j}$*,*
– $k \geq k_0$ *where* k_0 *only depends on the choice of the data* $\{(\tau_i, r_i)_{i=1,..,m}, c\}$*,*
– Γ *is an* $\widetilde{h}$*-hermitian smooth endomorphism such that* $\mathrm{tr}(\Gamma) = O(k^n)$*,*
then there exists for all $0 \leq \rho \leq \rho_0$ *a smooth solution* η_ρ *of*

$$\mathcal{B}_{\widetilde{h},\rho}(\eta_\rho) = \Gamma. \tag{18}$$

Proof We will use a continuity method on the Banach space $End(\mathcal{F})^{l,\alpha}$ with respect to the parameter ρ. First of all, for $\rho = 0$, one sees that it is possible to solve (18) by choosing

$$\eta_0 = \left(\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}\right)^{-1}\Gamma.$$

Therefore, if one denotes $I \subset \mathbb{R}_+$ the interval such that $\rho \in I$ if η_ρ is solution of (18), we have just proved that $I \neq \emptyset$. Apply the Implicit function Theorem to see that I is open. By Lemma 16, we know that the differential in η of $\mathcal{B}_{\mathfrak{q},\widetilde{h},\rho}$ is given by,

$$D_\eta\mathcal{B}_{\mathfrak{q},\widetilde{h},\rho}(\varsigma) = \left(\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}\right)\varsigma + \frac{\rho}{k}\Pi^{\mathscr{F}\otimes L^k,\boldsymbol{\tau}}_{\widetilde{h}\cdot\eta}(\varsigma).$$

But, from another side, we know that

$$|||\Pi^{\mathscr{F}\otimes L^k,\boldsymbol{\tau}}_{\widetilde{h}\cdot\eta}||| \leq \sum_i \tau_i r_i(r - r_i).$$

Now, with our choice of ρ_0 we get that $\mathcal{B}_{\mathfrak{q},\widetilde{h},\rho}$ is invertible since we have $\mathrm{tr}(\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}) = O(k^n)$.

Finally, we prove that I is closed. If one has a solution η of

$$\mathcal{B}_{\mathfrak{q},\widetilde{h},\rho}(\eta) = \Gamma, \tag{19}$$

then for all $U \in \mathcal{F} \otimes L^k_{|p}$,

$$\sum_i \langle s_i, U\rangle_{\widetilde{h}\cdot\eta}\langle U, s_i\rangle_{\widetilde{h}\cdot\eta} + \rho \sum_j \frac{\tau_j}{k}\langle U, \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h}\cdot\eta,j} U\rangle_{\widetilde{h}\cdot\eta} = \langle U, \Gamma U\rangle_{\widetilde{h}\cdot\eta}. \tag{20}$$

Since ρ and the τ_i are non negative, we immediately get that

$$\sum_i \langle s_i, U\rangle_{\widetilde{h}\cdot\eta}\langle U, s_i\rangle_{\widetilde{h}\cdot\eta} \leq \langle U, \Gamma U\rangle_{\widetilde{h}\cdot\eta}.$$

Moreover, since we have $\mathrm{tr}(\Gamma) = O(k^n)$ and $\frac{\rho}{k} = O(k^{n-1})$, there exists a constant $c'(k_0)$ such that

$$\sum_i \langle s_i, U\rangle_{\widetilde{h}\cdot\eta}\langle U, s_i\rangle_{\widetilde{h}\cdot\eta} \geq \frac{1}{1+c'}\langle U, \Gamma U\rangle_{\widetilde{h}\cdot\eta}.$$

Now, if one considers $\lambda_{max} \geq 0$ the maximal eigenvalue of η at p and v an associated eigenvector, we obtain that

$$\frac{1}{1+c'}\langle v, \Gamma v\rangle_{\widetilde{h}} \leq \lambda_{max} \sum_i |\langle v, s_i\rangle_{\widetilde{h}}|^2 \leq \langle v, \Gamma v\rangle_{\widetilde{h}}. \tag{21}$$

Since the $(s_i)_{i=1,..,N}$ form a free family, one gets that λ_{max} belongs to a compact set in C^0 norm and also the solution η of the considered equation, which is a definite positive hermitian endomorphism. With Lemma 16 we can see that differentiating (19), we get

$$\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}\partial\eta + \frac{\rho}{k}\Pi^{\mathscr{F}\otimes L^k,\boldsymbol{\tau}}_{\widetilde{h}\cdot\eta,i}\partial\eta = \partial\Gamma - \partial\left(\sum_i s_i\langle ., s_i\rangle_{\widetilde{h}}\right)\eta. \tag{22}$$

Since Γ is smooth and M compact, one gets from the C^0 estimate of η, a C^0 bound on $\partial\eta$, and by a similar way a C^0 bound on $\overline{\partial}\eta$. By differentiating (22), one sees again that $\overline{\partial}\partial\eta$ is bounded in C^0 norm and consequently, our solution η is bounded in $C^{1,1}$ norm. Finally, with Arzela-Ascoli's Theorem we get I closed and therefore $I = [0, \epsilon_k]$. □

Proposition 7 *Let us fix an integer $q \geq 1$. To each almost balanced metric $h_{k,q}$ corresponds a zero $\widetilde{h_q}$ of the moment map $\mu_{\mathcal{G}}$ on $\mathcal{Q}_0$ such that*

$$\|h_{k,q} - \widetilde{h_q}\|_{C^\alpha} = O\left(\frac{1}{k^{q-1-\alpha}}\right).$$

Moreover, we have the decomposition

$$\int_M \langle s_i, s_j\rangle_{\widetilde{h_q}}\frac{\omega^n}{n!} = \delta_{ij} + \boldsymbol{\eta}_{\widetilde{h_q}}$$

where $\boldsymbol{\eta}_{\widetilde{h_q}}$ is a matrix $N \times N$ such that

$$|||\boldsymbol{\eta}_{\widetilde{h_q}}||| = O\left(\|\boldsymbol{\sigma}_q(k)\|_{C^0}\right),$$

where $\boldsymbol{\sigma}_q(k)$ is given by Proposition 6.

Proof Indeed, by Proposition 6, there exists a metric $h_{k,q} \in Met(\mathcal{F} \otimes L^k)$ such that

$$\sum_i s_i \langle ., s_i \rangle_{h_{k,q}} = \frac{N + \epsilon_k V \sum_j \varepsilon_j r_j}{rV} Id + \boldsymbol{\sigma}_q(k) - \epsilon_k \sum_j \varepsilon_j \pi_{h_{k,q}}^{\mathscr{F} \otimes L^k}, \tag{23}$$

with $\|\boldsymbol{\sigma}_q(k)\|_{C^{\alpha+2}} \leq C_{q,\alpha} k^{n-q-1}$, $(s_i)_{i=1,..,N}$ a $Hilb_\omega(h_{k,q})$-orthonormal basis and still $\varepsilon_j = \frac{\tau_j}{k}$. Consider the metric $\widetilde{h_q} = h_{k,q}(\eta.,.)$ and a point $(s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Q}_0$. By perturbation of our almost balanced metric, we can get a zero of the moment map $\mu_{\mathcal{G}}$ on $\mathcal{Q}_0$. Indeed, by Lemma 7 with the data

$$\Gamma = \frac{N + \epsilon_k V \sum_j \varepsilon_j r_j}{rV} Id = \mathbf{O}(k^n) \qquad \widetilde{h} = h_{k,q} \qquad \rho = \epsilon_k,$$

we see that for k fixed such that $k \geq k_0$ we can find a smooth solution η of (18). Then, the metric $\widetilde{h_q} = h_{k,q} \cdot \eta$ is a zero of the moment map $\mu_{\mathcal{G}}$.

Now the fact that $\widetilde{h_q} = h_{k,q} \cdot \eta$ is close to $h_{k,q}$ is a consequence of the relation (20) which gives us for $\rho = \epsilon_k$ and v an eigenvector associated to the eigenvalue $\lambda \geq 0$ of η,

$$\begin{aligned}
(\lambda - 1) {\textstyle\sum_i} |\langle v, s_i \rangle_{h_{k,q}}|^2 &= \langle v, \Gamma v \rangle_{h_{k,q}} - \epsilon_k {\textstyle\sum_j} \varepsilon_j \langle v, \pi_{\widetilde{h_q},j}^{\mathscr{F} \otimes L^k} v \rangle_{h_{k,q}} \\
&\quad - {\textstyle\sum_i} |\langle v, s_i \rangle_{h_{k,q}}|^2, \\
&= \langle v, \Gamma v \rangle_{h_{k,q}} - \epsilon_k {\textstyle\sum_j} \varepsilon_j \langle v, \pi_{h_{k,q} \cdot (Id + (\eta - Id)),j}^{\mathscr{F} \otimes L^k} v \rangle_{h_{k,q}} \\
&\quad - {\textstyle\sum_i} |\langle v, s_i \rangle_{h_{k,q}}|^2, \\
&= \langle v, \Gamma v \rangle_{h_{k,q}} - \epsilon_k {\textstyle\sum_j} \varepsilon_j \langle v, \pi_{h_{k,q},j}^{\mathscr{F} \otimes L^k} v \rangle_{h_{k,q}} \\
&\quad - \langle v, \frac{1}{k} \Pi_{h_{k,q}}^{\mathscr{F},\tau} (\eta - Id) v \rangle_{h_{k,q}} \\
&\quad + \langle v, \frac{1}{k} \vartheta(Id - \eta) v \rangle_{h_{k,q}} - {\textstyle\sum_i} |\langle v, s_i \rangle_{h_{k,q}}|^2,
\end{aligned} \tag{24}$$

using Lemma 16. Here $\vartheta(Id - \eta)$ is an endomorphism of $\mathcal{F} \otimes L^k$ such that $\|\vartheta(Id - \eta)\|_{C^0} = O(\|Id - \eta\|_{C^0}^2)$. From another side, we get that

$$\epsilon_k \left\| \frac{1}{k} \Pi_{h_{k,q}}^{\mathscr{F},\tau} (Id - \eta) \right\|_{C^0} \leq \frac{\epsilon_k}{k} \left({\textstyle\sum_j} \tau_j r_j (r - r_j) \right) \|\eta - Id\|_{C^0}. \tag{25}$$

Since $h_{k,q}$ satisfies equation (23) by definition, we get from (25) and (24) that for some constants c_0, c_0', c_0'', independent of k,

$$\begin{aligned}
\|\eta - Id\|_{C^0} \left(1 - \frac{c_0 \epsilon_k}{k^{n+1}} \right) - \frac{c_0' \epsilon_k}{k^{n+1}} \|\eta - Id\|_{C^0}^2 &\leq \frac{c_0''}{k^n} \left(\|\Gamma - \Gamma\|_{C^0} + \|\boldsymbol{\sigma}_q(k)\|_{C^0} \right), \\
&\leq \frac{c_0''}{k^n} \|\boldsymbol{\sigma}_q(k)\|_{C^0}.
\end{aligned}$$

We obtain the expected estimate for k sufficiently large. Finally, since the s_i are orthonormal respectively to $Hilb_\omega(h_{k,q})$, we have

$$\int_M \langle s_i, s_j\rangle_{\widetilde{h_q}} \frac{\omega^n}{n!} = \int_M \langle s_i, s_j\rangle_{h_{k,q}} \frac{\omega^n}{n!} + \int_M \langle (\eta - Id)s_i, s_j\rangle_{h_{k,q}} \frac{\omega^n}{n!},$$
$$= \delta_{ij} + \int_M \langle (\eta - Id)s_i, s_j\rangle_{h_{k,q}} \frac{\omega^n}{n!}.$$

We use now the first part of the proof and Cauchy-Schwartz inequality. □

4.6 Explicit formulas and analytic estimates

Fix q a positive integer. From the almost balanced metrics $h_{k,q}$, we have just got a point z in the symplectic quotient $\mathcal{Z} = \mathcal{Q}_0//\mathcal{G}$, with a metric $\langle .,. \rangle = \widetilde{h_q} \in Met(\mathcal{F} \otimes L^k)$ which satisfies

$$\sum_{i=1}^{N} s_i \langle ., s_i\rangle_{\widetilde{h_q}} + \epsilon_k \sum_{j=1}^{m-1} \varepsilon_j \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},j} = Cst_k Id, \tag{26}$$

where $\mathfrak{q} = (s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Q}_0$ is a lifting of z. We now study the $SL(N)$-orbit of the point z. We are looking for an estimate of the quantity $\boldsymbol{\Lambda}_z$ in order to apply Proposition 5. We will apply the following lemma inspired from [16, Lemma 18] in order to ease the computation of $\boldsymbol{\Lambda}_z$.

Lemma 8 *Let $\widehat{\nu}_{\widehat{z}}^{\mathcal{Q}_0,SU(N)} : \mathfrak{su}(N) \to T_{\widehat{z}}\mathcal{Q}_0$ and $\widehat{\nu}_{\widehat{z}}^{\mathcal{Q}_0,\mathcal{G}} : Lie(\mathcal{G}^{\mathbb{C}}) \to T_{\widehat{z}}\mathcal{Q}_0$ be the infinitesimal actions induced by $SU(N)$ and $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{Q}_0$ and let $z \in \mathcal{Z}$ be represented by $\widehat{z} \in \mathcal{Q}_0$. Then for all $\xi \in \mathfrak{su}(N)$,*

$$\left\langle \mathfrak{q}_z^{SU(N)}(\xi), \xi \right\rangle = \left| \underline{\pi}\left(\widehat{\nu}_{\widehat{z}}^{\mathcal{Q}_0,SU(N)}(\xi)\right)\right|^2,$$

with $\underline{\pi} : T_{\widehat{z}}\mathcal{Q}_0 \to T_{\widehat{z}}\mathcal{Q}_0$ orthogonal projection onto $\mathrm{Im}\left(\widehat{\nu}_{\widehat{z}}^{\mathcal{Q}_0,\mathcal{G}}\right)^{\perp}$. In particular,

$$\boldsymbol{\Lambda}_z = \left(\min_{\xi\in\mathfrak{su}(N)} \frac{\left|\underline{\pi}\left(\widehat{\nu}_{\widehat{z}}^{\mathcal{Q}_0,SU(N)}(\xi)\right)\right|}{|\xi|} \right)^{-2}.$$

Consider the matrix

$$\mathsf{A} = (a_{ij})_{ij} \in \sqrt{-1}\mathfrak{su}(N)$$

and let

$$\sigma_i = \sum_{j=1}^{N} a_{ij} s_j, \qquad i = 1, ..., N$$

be the induced basis by the infinitesimal action of A. We look for the projection of

$$\underline{\sigma} = (\sigma_1, ..., \sigma_N, 0, ..., 0)$$

from the space $C^\infty(M, \mathcal{F}\otimes L^k)^N \times \prod_i C^\infty(M, \theta_i^* TGr(r_i, \mathcal{F}))$ onto the orthogonal complement of the subspace

$$\mathcal{P} = \left\{ \left(\mathfrak{g}s_1, ..., \mathfrak{g}s_N, (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},1})\mathfrak{g}\pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},1}, ..., (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},m-1})\mathfrak{g}\pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},m-1} \right), \atop \text{s.t. } \mathfrak{g} \in Lie(\mathcal{G}^{\mathbb{C}}) \right\}$$

which is the image of the infinitesimal action of $\mathcal{G}^{\mathbb{C}}$ of the point $\mathfrak{q}$, since $\theta_i^* TGr(r_i, \mathcal{F}) \simeq Hom(\mathrm{Im}(\theta_i), \mathrm{Im}(\theta_i)^{\perp_{\widetilde{h_q}}})$.

Notation We set as self-adjoint operator on $End(\mathcal{F}\otimes L^k)$,

$$\mathscr{B} : X \mapsto \frac{\epsilon_k}{Cst_k} \sum_{j=1}^{m-1} \varepsilon_j \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},j} X \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},j}.$$

If the condition

$$\frac{\epsilon_k}{Cst_k} \sum_{j=1}^{m-1} \varepsilon_j < 1 \tag{27}$$

holds, we notice that it makes sense to consider the operator $(Id - \mathscr{B})^{-1}$.

Lemma 9 *With the previous notations, assume that one has $Cst_k = O(k^n)$ in equation (26). Let*

$$\begin{aligned} B_{\mathsf{A}} &= (Id - \mathscr{B})^{-1} \left(\frac{1}{Cst_k} \textstyle\sum_{i,j} a_{ji} s_i \langle ., s_j \rangle \right) \\ &= \frac{1}{Cst_k} \textstyle\sum_{i,j} a_{ji} s_i \langle ., s_j \rangle + \frac{\epsilon_k}{(Cst_k)^2} \sum_l \varepsilon_l \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},l} \left(\sum_{i,j} a_{ji} s_i \langle ., s_j \rangle \right) \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},l} + .. \end{aligned}$$

be the hermitian endomorphism of $End(\mathcal{F}\otimes L^k)$ induced by the matrix A. Then the orthogonal projection of $\underline{\sigma}$ onto $\mathcal{P}$ is

$$\underline{p} = \left(B_{\mathsf{A}} s_1, .., B_{\mathsf{A}} s_N, (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},1}) B_{\mathsf{A}} \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},1}, .., (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},m-1}) B_{\mathsf{A}} \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},m-1} \right).$$

Proof We need to prove that for all $\mathfrak{g} \in Lie(\mathcal{G}^{\mathbb{C}})$,

$$\sum_i \langle B_{\mathsf{A}} s_i - \sigma_i, \mathfrak{g}s_i \rangle + \epsilon_k \sum_i \left\langle \varepsilon_i (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i}) B_{\mathsf{A}} \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i}, (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i}) \mathfrak{g} \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i} \right\rangle = 0$$

which is equivalent to saying that

$$\sum_i B_{\mathsf{A}} s_i \otimes s_i^* - \sigma_i \otimes s_i^* + \epsilon_k \sum_i \varepsilon_i (Id - \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i}) B_{\mathsf{A}} \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i} = 0,$$

i.e.

$$(Id - \mathscr{B})B_{\mathsf{A}} = \frac{1}{Cst_k}\sum_i \sigma_i \otimes s_i^*,$$

since one considers a point of $\mu_{\mathcal{G}}^{-1}(Cst \times Id)$ and therefore we have a determined metric via (26). The fact that

$$\varepsilon_i \epsilon_k = \frac{\tau_i k^{n-1}}{k^n} \frac{\chi(\mathcal{F}\otimes L^k)}{Vr - V\sum_j \frac{\tau_j k^{n-1}}{k^n}} = O(k^{n-1})$$

allows us to reach our conclusion since condition (27) is satisfied for k large. □

We set

$$\begin{aligned} \psi_i &= \sigma_i - B_{\mathsf{A}} s_i & 1 \leq i \leq N, \\ \psi_{N+i} &= (\pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i} - Id)B_{\mathsf{A}}\pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},i} & 0 \leq i \leq m-1, \\ \underline{\psi} &= (\psi_1, ..., \psi_{N+m-1}). \end{aligned}$$

Notation In all the following, we will denote

$$\left\|\underline{\psi}\right\|^2_{L^2(\omega)} = \sum_{i=1}^{N+m-1} ||\psi_i||^2_{Hilb_\omega(\widetilde{h_q})} = \sum_i \int_M |\psi_i|^2_{\widetilde{h_q}} \frac{\omega^n}{n!},$$

and $|||\mathsf{Q}|||^2_{L^2(\omega)}$ will stand for the L^2 operator norm induced by ω and the metric $\widetilde{h_q}$ of an endomorphism $\mathsf{Q} \in C^\infty(M, End(\mathcal{F}\otimes L^k))$. In particular, we can write the quantities $\Lambda_z, \boldsymbol{\Lambda}_z$ as

$$\Lambda_z^{-1} = \min_{i\mathsf{A}\in\mathfrak{su}(N), ||\mathsf{A}||=1} \left\|\underline{\psi}\right\|^2_{L^2(\omega)}, \quad \boldsymbol{\Lambda}_z^{-1} = \min_{i\mathsf{A}\in\mathfrak{su}(N), |||\mathsf{A}|||=1} \left\|\underline{\psi}\right\|^2_{L^2(\omega)}. \tag{28}$$

We also need the following definitions.

Definition 16 *Let $h_{\mathcal{F}}$ be a hermitian metric that we fix as a reference metric on $\mathcal{F}$ and fix an integer $\alpha > 2$. To the integer k, we associate the metric*

$$\widetilde{h_{\mathcal{F}}} = h_{\mathcal{F}} \otimes h_L^k$$

on $\mathcal{F}\otimes L^k$. We will say that for $R > 0$, another hermitian metric $\widetilde{h_1} = h_1 \otimes h_L^k$ on $\mathcal{F}\otimes L^k$ constructed in a similar way has (R,α)-bounded geometry if the two following conditions are satisfied:

$$\widetilde{h_1} > \frac{1}{R}\widetilde{h_{\mathcal{F}}}, \qquad \left\|\widetilde{h_1} - \widetilde{h_{\mathcal{F}}}\right\|_{C^\alpha} < R,$$

where $\|.\|_{C^\alpha}$ designs the standard norm C^α determined by the reference metric $\widetilde{h_{\mathcal{F}}}$. These conditions are equivalent to

$$h_1 > \frac{1}{R}h_{\mathcal{F}}, \quad \|h_1 - h_{\mathcal{F}}\|_{C^\alpha(h_1)} < k^{\alpha/2}R.$$

Clearly, up to a modification of R, this definition is independent of the choice of the metric $h_{\mathcal{F}}$.

Definition 17 *Consider a point $(s_0, ..., s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Q}_0$ and R a positive real. We will say that the basis $(s_i)_{i=1,..,N}$ has (R,α)-bounded geometry if there exists a smooth hermitian metric $\widetilde{h}_q$ which satisfies the condition (26) and has (R,α)-bounded geometry.*

We are ready to introduce the main result of this subsection.

Theorem 4 *Let $\mathscr{F}$ be a simple holomorphic filtration. For all $R > 0$, there exist some constants $C = C(R, h_{\mathcal{F}}, h_L)$ and $\varepsilon(R, h_{\mathcal{F}}, h_L) < \frac{1}{2}$ such that if, for all k, the basis $(s_i)_{i=1..N} \in H^0(M, \mathcal{F} \otimes L^k)$ satisfying (29) has (R,α)-bounded geometry with $|||\boldsymbol{\eta}||| < \varepsilon$ and $Cst_k = O(k^n)$, then for all matrix $\mathsf{A} = (a_{ij})_{ij} \in \sqrt{-1}\mathfrak{su}(N)$, we have*

$$\|\mathsf{A}\| \leq Ck \left\|\underline{\psi}\right\|_{L^2(\omega)},$$

where $\underline{\psi} \in \mathcal{P}^{\perp}$ is the orthogonal projection to $\mathcal{P}$ of $\underline{\sigma}$. For the corresponding point $z \in \mathcal{Z}$, we have

$$\Lambda_z \leq C^2k^2, \qquad \boldsymbol{\Lambda}_z \leq C^2k^2.$$

We shall postpone the proof of the theorem and introduce now some useful identities and lemmas.

First of all, we set the following decomposition with respect to the metric satisfying equation (26),

$$\langle s_i, s_j \rangle_{L^2(\omega)} = \int_M \langle s_i, s_j \rangle \frac{\omega^n}{n!} = \delta_{ij} + \boldsymbol{\eta}_{ij}, \tag{29}$$

where $\boldsymbol{\eta} = (\boldsymbol{\eta}_{ij})$ is a trace free hermitian matrix $N \times N$. Thus, $\boldsymbol{\eta} \equiv 0$ if and only if $\mathscr{F}$ is k-balanced.

We will also use the following well-known facts (for all hermitian matrices S, R of size $N \times N$).

$$|||\mathsf{R}||| \leq ||\mathsf{R}|| \leq \sqrt{N}\, |||\mathsf{R}|||\,, \tag{30}$$

$$|\mathtt{tr}(\mathsf{SRS})| \leq \|\mathsf{S}\|^2\, |||\mathsf{R}|||\,, \tag{31}$$

$$|\mathtt{tr}(\mathsf{RS})| \leq \sqrt{N}\, \|\mathsf{S}\|\, |||\mathsf{R}|||\,. \tag{32}$$

Lemma 10 *Under previous assumptions, if the basis of holomorphic sections $(s_i)_{i=1..N} \in H^0(M, \mathcal{F} \otimes L^k)$ has (R,α)-bounded geometry, and $|||\boldsymbol{\eta}||| < \frac{1}{2}$, then*

$$||\mathsf{A}||^2 \leq 2\left(|||B_{\mathsf{A}}|||^2_{L^2(\omega)} + \left\|\underline{\psi}\right\|^2_{L^2(\omega)}\right).$$

Proof To prove the second inequality, we use the fact that we have an orthogonal decomposition of $C^\infty(M, \mathcal{F} \otimes L^k)^N \times \prod_{i=1}^{m-1} C^\infty(M, \theta_i^* TGr(r_i, \mathcal{F}))$:

$$\underline{\sigma} = \underline{\psi} + \underline{p},$$

with $\underline{p} \in \mathcal{P}$, $\underline{\psi} \in \mathcal{P}^{\perp}$. But,

$$\|\underline{\sigma}\|^2_{L^2(\omega)} = \sum_{i,j} |a_{ij}|^2 + \sum_{i,j,l} a_{ij}\boldsymbol{\eta}_{jl}a_{li} = \|\mathsf{A}\|^2 + \mathrm{tr}(\mathsf{A}\boldsymbol{\eta}\mathsf{A}),$$

and since $|||\boldsymbol{\eta}||| < \frac{1}{2}$ we have by (31) that

$$||\mathsf{A}||^2 < 2\,||\underline{\sigma}||^2_{L^2(\omega)}\,,$$

and thus

$$\frac{1}{2}||\mathsf{A}||^2 \leq \left\|\underline{\psi}\right\|^2_{L^2(\omega)} + \left\|\underline{p}\right\|^2_{L^2(\omega)} \leq \left\|\underline{\psi}\right\|^2_{L^2(\omega)} + |||B_{\mathsf{A}}|||^2_{L^2(\omega)}\,.$$

□

We use Poincaré inequality to evaluate the term $|||B_{\mathsf{A}}|||^2_{L^2(\omega)}$.

Lemma 11 *Assume that the holomorphic filtration $\mathscr{F}$ is simple and that $\mathsf{A} \in \sqrt{-1}\mathfrak{su}(N)$. If the basis $(s_i)_{i=1..N}$ of $H^0(M, \mathcal{F} \otimes L^k)$ has (R,α)-bounded geometry, then there exist two constants C_1, C_2 that depend only on R and of the reference metric $h_{\mathcal{F}}$ on $\mathcal{F}$ such that for k large enough,*

$$|||B_{\mathsf{A}}|||^2_{L^2(\omega)} \leq C_1 \left\|\overline{\partial} B_{\mathsf{A}}\right\|^2_{L^2(\omega)} + C_2\left(|||\boldsymbol{\eta}||| + \frac{1}{k}\right)^2 \|\mathsf{A}\|^2\,.$$

Proof The fact that $\mathscr{F}$ is simple implies that for any $\gamma > 1$, there exists $c(h_{\mathcal{F}}, \gamma)$ such that if $h \in Met(\mathcal{F})$ is a metric satisfying

$$\gamma h_{\mathcal{F}} > h > \frac{1}{\gamma} h_{\mathcal{F}},$$

then

$$||\varpi||^2_{L^2(\omega)} \leq c\left\|\overline{\partial}\varpi\right\|^2_{L^2(\omega)} + \frac{1}{rV} \mid \int_M \mathtt{tr}(\varpi) dV \mid^2$$

for all $\varpi \in End(\mathcal{F})$ such that $\varpi(\mathcal{F}_i) \subset \mathcal{F}_i$. This is simply the Poincaré inequality with respect to the metric ω (see also [16, Lemma 25]) whose volume is V. Now, since B_{A} is hermitian, we can decompose B_{A} under the form

$$B_{\mathsf{A}} = \mathsf{T}_{B_{\mathsf{A}}} + \mathsf{D}_{B_{\mathsf{A}}} + \mathsf{T}^*_{B_{\mathsf{A}}},$$

where $\mathsf{T}_{B_{\mathsf{A}}}$ is upper triangular and $\mathsf{D}_{B_{\mathsf{A}}}$ diagonal. Let $\Pi(B_{\mathsf{A}}) = \mathsf{T}_{B_{\mathsf{A}}} + \frac{1}{2}\mathsf{D}_{B_{\mathsf{A}}}$. Then $\Pi(B_{\mathsf{A}})$ is an endomorphism of $\mathcal{F}$ such that $\Pi(B_{\mathsf{A}})(\mathcal{F}_i) \subset \mathcal{F}_i$. Therefore,

$$||\Pi(B_{\mathsf{A}})||^2_{L^2(\omega)} \leq C_2\left\|\overline{\partial}\Pi(B_{\mathsf{A}})\right\|^2_{L^2(\omega)} + \frac{1}{rV}\left(\int_M \frac{1}{2}\mathtt{tr}(B_{\mathsf{A}})\frac{\omega^n}{n!}\right)^2.$$

Nevertheless, the fact that B_{A} is hermitian also gives us that

$$||\Pi(B_{\mathsf{A}})||^2_{L^2(\omega)} = \frac{1}{2}\,||B_{\mathsf{A}}||^2_{L^2(\omega)}$$

and $\left\|\overline{\partial}\Pi(B_{\mathsf{A}})\right\|^2_{L^2(\omega)} = \frac{1}{2}\left\|\overline{\partial}B_{\mathsf{A}}\right\|^2_{L^2(\omega)}$. Thus, we have

$$|||B_{\mathsf{A}}|||^2_{L^2(\omega)} \leq ||B_{\mathsf{A}}||^2_{L^2(\omega)} \leq C_1\left\|\overline{\partial}B_{\mathsf{A}}\right\|^2_{L^2(\omega)} + \frac{1}{rV}\left(\int_M \mathtt{tr}(B_{\mathsf{A}})\frac{\omega^n}{n!}\right)^2.$$

Now, A is trace free and $\frac{1}{Cst_k}\int_M \mathrm{tr}(\sum_i s_i\langle ., s_i\rangle)dV = O(1)$. Therefore we notice with (32) that for k large enough, there exists c such that

$$\int_M \mathtt{tr}(B_{\mathsf{A}})\frac{\omega^n}{n!} \leq \frac{\sum_{i,j} a_{ij}\boldsymbol{\eta}_{ij}}{Cst_k} + \sum_{p=1}^{\infty}\frac{1}{k^p}\left(\frac{\epsilon_k}{Cst_k}\right)^p \sum_{l=1}^{m-1} c\tau_l^p r_l \|\mathsf{A}\|^p,$$
$$\leq C_2\left(|||\boldsymbol{\eta}||| + \frac{1}{k}\right)\|\mathsf{A}\|$$

for $C_2 \geq 1$ large enough, and we get the expected estimate. □

To get rid of the term $\left|\left|\left|\overline{\partial} B_{\mathsf{A}}\right|\right|\right|^2_{L^2(\omega)}$, we need the following lemmas.

Lemma 12 *Under previous assumptions, there exist two independent constants $C^{(1)}(j_0)$, $C^{(2)}(j_0)$ such that for all $j \leq j_0$,*

$$\sum_i \left|\nabla^j s_i(z)\right|^2 \leq C^{(1)} k^{j+n} \text{ for all } z \in M,$$
$$\left\|\nabla^j B_{\mathsf{A}}\right\|^2_{L^2(\omega)} \leq C^{(2)} k^j ||\mathsf{A}||^2.$$

Proof We have pointwise the following Poincaré inequality for any holomorphic section f,

$$\left|\nabla^j f(x)\right|_h^2 \leq c\int_{B(x)} |f|_h^2 \frac{\omega^n}{n!},$$

and for $\widetilde{h_q}$,

$$\left|\nabla^j f(x)\right|^2_{\widetilde{h_q}} \leq ck^j \int_{B(x)} |f|^2_{\widetilde{h_q}} \frac{\omega^n}{n!} \leq ck^j\int_M |f|^2_{\widetilde{h_q}}\frac{\omega^n}{n!},$$

where $B(x)$ is a geodesic ball centered at $x \in M$ and c depends only on R and ω. Now, we sum up for all i and use the fact that

$$\sum_i |s_i(x)|^2 \leq \mathrm{tr}\left(\textstyle\sum_i s_i(x)\langle ., s_i(x)\rangle\right) + \mathrm{tr}\left(\epsilon_k \textstyle\sum_j \varepsilon_j \pi^{\mathscr{F}\otimes L^k}_{\widetilde{h_q},j}\right) \leq rCst_k.$$

Hence, we get the inequality since $Cst_k = O(k^n)$.
Moreover, since B_{A} is not holomorphic, we look at $M' = M \times \overline{M}$ (cf. [16, p. 507]) where $\overline{M}$ is equipped with the opposite complex structure and p_1, p_2 are the projections on the first and second factor. To the connection and the metric on $L \to M$ correspond a connection and a metric on $\overline{L} \to \overline{M}$ equipped with the opposite complex structure. Let $\mathcal{F}' \to M'$ be defined by $p_1^*\left(\overline{\mathcal{F}} \otimes \overline{L}^k\right)^\vee \otimes p_2^*\left(\mathcal{F}\otimes L^k\right)$. Let $s^\vee \in H^0\left(\overline{M}, \overline{\mathcal{F}}\otimes\overline{L}^k\right)^\vee$ be the holomorphic section associated to $s \in H^0(M, \mathcal{F}\otimes L^k)$ via the C^∞ isomorphism of bundle defined by the metric. Then, we set

$$\widetilde{B_{\mathsf{A}}} = (Id - \mathscr{B}^{-1})\left(\frac{1}{Cst_k}\sum_{i,j} a_{ji} s_i \otimes s_j^\vee\right),$$

which is a holomorphic section of $\mathcal{F}'$. Thus, if we denote $\Sigma_{\mathsf{A}} = \sum_{i,j} a_{ji} s_i \langle ., s_j \rangle$ we notice by Cauchy-Schwarz inequality that

$$\langle \Sigma_{\mathsf{A}}, \mathscr{B}^p(\Sigma_{\mathsf{A}}) \rangle \leq \|\Sigma_{\mathsf{A}}\| \|\mathscr{B}^p(\Sigma_{\mathsf{A}})\| \leq \left(\frac{\epsilon_k \sum_i \varepsilon_i}{Cst_k} \right)^p \|\Sigma_{\mathsf{A}}\|^2.$$

Then, we get

$$\begin{aligned} \left\| \widetilde{B_{\mathsf{A}}} \right\|^2_{L^2(\omega)} &= \langle \Sigma_{\mathsf{A}} + \mathscr{B}(\Sigma_{\mathsf{A}}) + \mathscr{B}^2(\Sigma_{\mathsf{A}}) + ..., \Sigma_{\mathsf{A}} + \mathscr{B}(\Sigma_{\mathsf{A}}) + \mathscr{B}^2(\Sigma_{\mathsf{A}}) + ... \rangle_{L^2(\omega)}, \\ &= \langle \Sigma_{\mathsf{A}}, \Sigma_{\mathsf{A}} \rangle_{L^2(\omega)} (1 + O(1/k)), \\ &= \frac{(1 + O(1/k))}{(Cst_k)^2} \int_M \sum_{i,j,k,l} a_{ij} a_{kl} \langle s_i, s_j \rangle \langle s_k, s_l \rangle \frac{\omega^n}{n!}, \\ &= \frac{(1 + O(1/k))}{(Cst_k)^2} \mathtt{tr} \left(\mathsf{A} \left(Id + \boldsymbol{\eta} \right) \left(Id + {}^t\overline{\boldsymbol{\eta}} \right) {}^t\overline{\mathsf{A}} \right). \end{aligned}$$

But, we know that $Cst_k = O(k^n)$. Since we have chosen a basis with (R, α)-bounded geometry, we obtain by inequality (31),

$$\left\| \widetilde{B_{\mathsf{A}}} \right\|_{L^2(\omega)} \leq C k^{-n} ||\mathsf{A}||.$$

Now, we notice that B_{A} is simply the restriction of $\widetilde{B_{\mathsf{A}}}$ over the diagonal of M'. We apply the inequality for a holomorphic section as in the first step of the proof, and get

$$\left\| \nabla^j \widetilde{B_{\mathsf{A}}} \right\|^2_{L^2(\omega)} \leq C^{(2)} k^j ||\mathsf{A}||^2,$$

which allows us to conclude. □

Lemma 13 *There exists a constant C_3 which depends only on R and of the reference metric $h_{\mathcal{F}}$ such that for k large enough, we have*

$$\sum_{i=1}^{N+m-1} ||\overline{\partial} \psi_i||^2_{L^2(\omega)} = |||\overline{\partial} B_{\mathsf{A}}|||^2_{L^2(\omega)} \leq k C_3 \left\| \underline{\psi} \right\|_{L^2(\omega)} ||\mathsf{A}||. \tag{33}$$

Proof We shall begin to prove the LHS equality. Pointwise, we have

$$\begin{aligned} \sum_i |\overline{\partial} \psi_i|^2 &= \sum_i \left| \overline{\partial}(B_{\mathsf{A}} s_i) \right|^2 + \sum_i \epsilon_k \varepsilon_i \left| \overline{\partial} \left((Id - \pi^{\mathscr{F} \otimes L^k}_{\widetilde{h_q}, i}) B_{\mathsf{A}} \pi^{\mathscr{F} \otimes L^k}_{\widetilde{h_q}, i} \right) \right|^2, \\ &= \sum_i \left| \overline{\partial}(B_{\mathsf{A}}) s_i \right|^2 + \sum_i \epsilon_k \varepsilon_i \left| (Id - \pi^{\mathscr{F} \otimes L^k}_{\widetilde{h_q}, i}) \overline{\partial}(B_{\mathsf{A}}) \pi^{\mathscr{F} \otimes L^k}_{\widetilde{h_q}, i} \right|^2, \end{aligned} \tag{34}$$

since the s_i and θ_i are holomorphic. The RHS of (34) is, by definition of the metric we have fixed, the operator norm $|||B_{\mathsf{A}}|||^2$ at the considered point p of M. By integration

over the manifold, we have the first part of the lemma.
Now, we have with previous equality,

$$|||\overline{\partial}B_{\mathsf{A}}|||^2_{L^2(\omega)} \leq \sqrt{\sum_{i=1}^{N} ||\Delta B_{\mathsf{A}} s_i||^2_{L^2(\omega)} \sum_{i=1}^{N} ||B_{\mathsf{A}} s_i||^2_{L^2(\omega)}}$$
$$+ \epsilon_k \sqrt{\sum_{j=1}^{m-1} \varepsilon_j^2 \left\| \Delta \left(B_{\mathsf{A}} \pi_{h,j}^{\mathscr{F}\otimes L^k} \right) \right\|^2_{L^2(\omega)} \sum_{j=1}^{m-1} \left\| B_{\mathsf{A}} \pi_{\widetilde{h_q},j}^{\mathscr{F}\otimes L^k} \right\|^2_{L^2(\omega)}}$$

and from another side,

$$\|\Delta (B_{\mathsf{A}} s_i)\|_{L^2(\omega)} \leq \left\|\nabla^2(B_{\mathsf{A}})\right\|_{L^2(\omega)} + \|2\nabla B_{\mathsf{A}} \cdot \nabla s_i\|_{L^2(\omega)},$$
$$\left\| \Delta(B_{\mathsf{A}} \pi_{\widetilde{h_q},j}^{\mathscr{F}\otimes L^k}) \right\|_{L^2(\omega)} \leq \left\|\nabla^2(B_{\mathsf{A}})\right\|_{L^2(\omega)} + \left\| 2\nabla B_{\mathsf{A}} \cdot \nabla \pi_{\widetilde{h_q},j}^{\mathscr{F}\otimes L^k} \right\|_{L^2(\omega)}.$$

We conclude using the last lemma (2^{nd} inequality) and the first part of the proof. □

Proof of Theorem 4. With the last lemmas, we have straightforward,

$$\|\mathsf{A}\|^2 \leq 2\left(|||B_{\mathsf{A}}|||^2_{L^2(\omega)} + \|\underline{\psi}\|^2_{L^2(\omega)} \right),$$
$$\leq 2\left(C_1 \left\|\overline{\partial}B_{\mathsf{A}}\right\|^2_{L^2(\omega)} + C_2 \left(|||\boldsymbol{\eta}||| + \frac{1}{k} \right)^2 \|\mathsf{A}\|^2 + \|\underline{\psi}\|^2_{L^2(\omega)} \right),$$
$$\leq C_4 \left(k \|\underline{\psi}\|_{L^2(\omega)} ||\mathsf{A}|| + \left(|||\boldsymbol{\eta}||| + \frac{1}{k} \right)^2 \|\mathsf{A}\|^2 + \|\underline{\psi}\|^2_{L^2(\omega)} \right).$$

With the assumption $|||\boldsymbol{\eta}||| < \varepsilon$ and k sufficiently large, we choose ε such that we have $C_4\left(\epsilon + \frac{1}{k}\right)^2 < \frac{1}{2}$. Then, there exists a constant C, independent of k, such that

$$\|\mathsf{A}\|^2 \leq C\left(k \|\underline{\psi}\|_{L^2(\omega)} ||\mathsf{A}|| + \|\underline{\psi}\|^2_{L^2(\omega)} \right).$$

If $k \|\underline{\psi}\|_{L^2(\omega)} ||\mathsf{A}|| \leq \|\underline{\psi}\|^2_{L^2(\omega)}$ then the result is clear since $k \geq 1$. Otherwise, after simplification, we get exactly the expected inequality.
The second part of the theorem is a consequence of the equality (28) and of the fact that $|||\mathsf{A}||| \leq ||\mathsf{A}||$. □

4.7 Approximation Theorem for $\boldsymbol{\tau}$-Hermite-Einstein metrics

In this section, we achieve the proof for Theorem 5 using the analytical estimate of the previous part and the construction of an almost balanced metric. We fix now as a reference metric the metric $h_{\mathcal{F}} = h_\infty$, which is a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric solution of equation (13).

For any trace free matrix $\mathsf{S} \in \sqrt{-1}\mathfrak{su}(N)$, we can consider the action of $SL(N)$ on $z \in \mathcal{Z}$ to obtain another point $e^{\mathsf{S}} * z$ of the symplectic quotient. This gives us a new hermitian metric $\widetilde{h_{\mathsf{S}}} \in Met(\mathcal{F} \otimes L^k)$ that still satisfies equation (26) and depends on q. Let $\boldsymbol{\eta}(\mathsf{S})$ be the matrix satisfying the decomposition (29) for this new hermitian metric $\widetilde{h_{\mathsf{S}}}$. Under these conditions and with the notations of Proposition 6, we have the following estimates.

Lemma 14 *Fix a real number $R > 0$ and $\mathsf{S} \in \sqrt{-1}\mathfrak{su}(N)$ with $|||\mathsf{S}||| \leq \frac{1}{2}$.*
1. If $q > \alpha + 1$, then there exists a constant C_5 (independent of k and R) such that if one has

$$|||\mathsf{S}||| + \frac{1}{k} \leq C_5 R,$$

then the metric $\widetilde{h_{\mathsf{S}}}$ has (R, α)-bounded geometry.
2. There exists a constant C_6 (independent of k) such that

$$|||\boldsymbol{\eta}(\mathsf{S})||| \leq C_6 \left(|||\mathsf{S}||| + ||\boldsymbol{\sigma}_q(k)||_{C^0}\right).$$

Proof First of all, the construction is invariant under $SU(N)$ action. So, we can assume that $\mathsf{S} = diag(\lambda_i)$ is a diagonal matrix with $\sum_i \lambda_i = 0$. We consider the two points $z = (s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Z}$ and $z' = (e^{\lambda_1} s_1, ..., e^{\lambda_N} s_N, A, \theta_1, ..., \theta_{m-1}) \in \mathcal{Z}$. From Lemma 7, there exists a smooth endomorphism $\eta_{\mathsf{S}} \in End(\mathcal{F} \otimes L^k)$ such that

$$\mathcal{B}_{\widetilde{h_q}, \epsilon_k}(\eta_{\mathsf{S}}) = C_k Id.$$

Fix $\widetilde{h_{\mathsf{S}}} = \widetilde{h_q} \cdot \eta_{\mathsf{S}}$. Now, by definition,

$$\sum_i s_i \langle ., s_i \rangle_{\widetilde{h_{\mathsf{S}}}} + \epsilon_k \sum_j \varepsilon_j \pi^{\mathscr{F} \otimes L^k}_{\widetilde{h_{\mathsf{S}}}, j} = Cst_k Id + \sum_i (1 - e^{2\lambda_i}) s_i \langle ., s_i \rangle_{\widetilde{h_{\mathsf{S}}}}.$$

We apply the same reasoning that in the proof of Proposition 7 using the estimate of Lemma 12 (1$^{\text{rst}}$ inequality). Thus, we have that

$$\left\| \widetilde{h_q} - \widetilde{h_{\mathsf{S}}} \right\|_{C^\alpha} \leq c_1 \, |||\mathsf{S}||| \, .$$

Moreover, by Proposition 7, the metric $\widetilde{h_q}$ differs from the reference metric $\widetilde{h_\infty}$ by a term of the form $O(1/k)$ in C^α norm. Hence $\left\| \widetilde{h_\infty} - \widetilde{h_{\mathsf{S}}} \right\|_{C^\alpha} < R$ by choosing suitably C_5. Finally, for C_5 chosen small enough, we can also ask that $\widetilde{h_{\mathsf{S}}} > \frac{1}{R} \widetilde{h_\infty}$ since the quantity $|||\mathsf{S}|||$ is bounded by assumption.
For the second assertion, we notice that

$$\begin{aligned} \boldsymbol{\eta}(\mathsf{S})_{ij} &= \int_M \left\langle \eta_{\mathsf{S}} e^{\lambda_i} s_i, e^{\lambda_j} s_j \right\rangle_{\widetilde{h_q}} dV - \delta_{ij}, \\ &= \int_M \left\langle (\eta_{\mathsf{S}} - Id) e^{\lambda_i} s_i, e^{\lambda_j} s_j \right\rangle_{\widetilde{h_q}} dV + \left(\int_M \left\langle e^{\lambda_i} s_i, e^{\lambda_j} s_j \right\rangle_{\widetilde{h_q}} dV - \delta_{ij} \right). \end{aligned} \tag{35}$$

From the first part of the proof and Proposition 7, the first term of the RHS of (35) is bounded in C^0 norm by a multiple of $|||\mathsf{S}|||^2$. By Proposition 7, the second term of the RHS is bounded by a multiple of $|||\mathsf{S}||| + ||\boldsymbol{\sigma}_q(k)||_{C^0}$. □

With the next proposition we check that all the assumptions of Proposition 5 are satisfied.

Proposition 8 *Let R and q be positive real numbers.*
If we assume $|||\mathsf{S}||| \leq \min\{\delta, \delta k^{n-q+1}\}$ with $\delta(R, M, \mathscr{F})$ small enough, then the metric $\widetilde{h_{\mathsf{S}}}$ has (R,α)-bounded geometry, the basis $(s_i)_{i=1,..,N}$ has (R,α)-bounded geometry and $z = (s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1})$ is a zero of the moment map $\mu_{\mathcal{G}}$. Moreover, $\mu_{SU(N)} = \boldsymbol{\eta}(\mathsf{S})$ where the matrix $\boldsymbol{\eta}(\mathsf{S})_{ij}$ satisfies (29) with

$$\|\boldsymbol{\eta}(\mathsf{S})\| = O(k^{3n/2-q-1}).$$

Proof With the choice of $|||\mathsf{S}||| + \frac{1}{k} \leq C_5 R$, Lemma 14 gives us that if $|||\mathsf{S}||| \leq \delta$ with

$$\delta = \min(C_5 R/2, 1/2),$$

then the metric h_{S} has (R,α)-bounded geometry. From the almost balanced metrics $h_{k,q}$ which satisfy

$$\sum_i s_i \langle ., s_i \rangle_{h_{k,q}} = C_k Id - \epsilon_k \sum_j \varepsilon_j \pi^{\mathscr{F}\otimes L^k}_{h_{k,q},j} + \boldsymbol{\sigma}_q(k), \tag{36}$$

and $\|\boldsymbol{\sigma}_q(k)\|_{C^{\alpha+2}} = O(k^{n-q-1})$ we can apply Proposition 7. Under these conditions, we obtain from inequality (30) and Lemma 14 that

$$\|\boldsymbol{\eta}(\mathsf{S})\| \leq \sqrt{N} |||\boldsymbol{\eta}(\mathsf{S})||| \leq C_6 k^{n/2}(\delta + C_7) k^{n-q-1}.$$

□

Finally, we get the main result of this paper.

Theorem 5 *Let $\mathscr{F}$ be an irreducible holomorphic filtration equipped with a $\boldsymbol{\tau}$-Hermite-Einstein metric h_{HE} on a smooth projective manifold. Then $\mathscr{F}$ is balanced and there exists a sequence of balanced metrics $\boldsymbol{h}_k$ which converges in C^∞ sense towards a metric h_∞ conformally $\boldsymbol{\tau}$-Hermite-Einstein, i.e. towards h_{HE} up to a conformal change.*

Proof We begin to prove that we can construct a sequence of balanced metrics which converges in C^α topology towards the conformally $\boldsymbol{\tau}$-Hermite-Einstein metric $\widetilde{h_\infty}$, solution of equation (13).

Let ε be fixed by Theorem 4 and δ by Proposition 8. Apply Proposition 8 with $R > 0$, $q > \frac{3n}{2} + 2 + \alpha$ and $\|\mathsf{S}\| \leq \min\{\delta k^{n-q+1}, \varepsilon\} \leq \delta$. We obtain a point $z \in \mathcal{Z}$, represented by $\{s_1, ..., s_N, A, \theta_1, ..., \theta_{m-1}\} \in \mathcal{Q}_0$. By Theorem 4, we know that at that point, $\boldsymbol{\Lambda}_z \leq C^2 k^2$. Again, by Proposition 8, we get

$$\lambda \|\boldsymbol{\eta}_{\mathsf{S}}\| \leq \lambda C_8 k^{3n/2-q-1} \leq C_9 k^{3n/2+1-q}. \tag{37}$$

Since $|||\mathsf{S}||| \leq ||\mathsf{S}||$, we want to apply Proposition 5 with the data

$$\mu_{SU(N)}(z_0) = \boldsymbol{\eta}_{\mathsf{S}}, \qquad \lambda = C^2 k^2,$$

and δ given by Proposition 8. But inequality (37) asserts that the quantity $\lambda\|\mu_{SU(N)}(z_0)\|$ can be chosen smaller than δ for k large enough since $q > \frac{3n}{2} + 2$. Then, by Proposition 5, we obtain that

$$|||\mathsf{S}||| \leq ||\mathsf{S}|| \leq C_9 k^{3n/2+1-q},$$

and also the existence of a metric $\widetilde{h_{\mathsf{S}}}$ close in C^0 topology of $\widetilde{h_\infty}$ up to an error in $O\left(k^{3n/2-q+1}\right)$ and k-balanced. In fact, we also have

$$\|h_{\mathsf{S}} - h_\infty\|_{C^\alpha} = O\left(k^{3n/2-q+1+\alpha}\right),$$

and consequently, the convergence in C^α topology towards h_∞ for all $\alpha > 2$. The theorem is proved using now Proposition 3. □

Corollary 1 *The Hermite-Einstein connection on the bundle $\mathcal{F}$ of the stable filtration $\mathscr{F}$ is unique up to an holomorphic automorphism of $\mathscr{F}$.*

5 Applications to the case of Vortex type equations

In this part, we give some applications of Theorem 5, and in particular we study the case of coupled Vortex equations for which we have found it impossible to develop a direct method. Instead of working with the projective manifold M, we consider the Vortex equations as $\boldsymbol{\tau}$-Hermite-Einstein equations on a higher dimensional manifold and apply a dimensional reduction procedure.

5.1 Equivariant holomorphic filtrations

Consider G a connected simply connected semisimple complex Lie group and $P \subset G$ a parabolic subgroup of G. In particular, G/P is a Flag manifold. We denote K the maximal compact subgroup of G and consider $X = M \times G/P$ with trivial action on M. The Kähler structure on M and G/P define a Kähler structure on X.

Remark 8 One could also have construct X as a projectively flat G/P-bundle. See [7] for details.

Definition 18 *We will say that a coherent sheaf $\mathcal{F}$ is G-equivariant if the action of G on X can be lifted holomorphically to $\mathcal{F}$. A G-equivariant filtration $\mathscr{F}$ on X is a filtration of G-invariant coherent subsheaves of a G-equivariant sheaf $\mathcal{F}$ on X,*

$$\mathscr{F} : 0 = \mathcal{F}_0 \hookrightarrow ... \hookrightarrow \mathcal{F}_m = \mathcal{F}.$$

If the sheaves $\mathcal{F}_i$ are locally free, we say that the filtration is holomorphic.

Definition 19 *A filtration $\mathscr{F}$ is G-equivariantly $\boldsymbol{\tau}$-stable (resp. semi stable) if $\mathscr{F}$ is G-equivariant and for any G-invariant proper subfiltration $\mathscr{F}' \hookrightarrow \mathscr{F}$, we have*

$$\mu_{\boldsymbol{\tau}}(\mathscr{F}') < \mu_{\boldsymbol{\tau}}(\mathscr{F}) \qquad (resp.\ \ \leq).$$

Moreover, a filtration is G-equivariantly polystable if it is a direct sum of $\boldsymbol{\tau}$-stable G-equivariant filtrations with same slope $\mu_{\boldsymbol{\tau}}$.

Definition 20 *A filtration $\mathscr{F}$ is said to be G-equivariantly Gieseker $\mathbf{R}$-stable (resp. semi-stable) if $\mathscr{F}$ is G-equivariant and for k large, one has for all proper G-invariant subfiltration $\mathscr{F}'$ of $\mathscr{F}$,*

$$\frac{\mathscr{P}_{\mathbf{R},\mathscr{F}'}(k)}{r(\mathcal{F}')} < \frac{\mathscr{P}_{\mathbf{R},\mathscr{F}}(k)}{r(\mathcal{F})} \qquad (resp.\ \leq).$$

Proposition 9 *Let $\mathscr{F}$ a G-equivariant holomorphic filtration on X. Then, $\mathscr{F}$ is G-equivariantly $\boldsymbol{\tau}$-stable (resp. Gieseker $\mathbf{R}$-stable) if and only if it is G-equivariantly indecomposable, and considered as a holomorphic filtration, has a direct sum decomposition into $\boldsymbol{\tau}$-stable (resp. Gieseker $\mathbf{R}$-stable) holomorphic filtrations $\mathscr{F}_j$, $1 \leq j \leq j_0$ such that these filtrations are images one from another by an element of G.*

Proof This is similar to [18, Theorem 6] and [1, Theorem 2.2] and thus will be omitted.

One obtains a HKDUY correspondence for the holomorphic G-equivariant filtrations.

Theorem 5.11 Let $\boldsymbol{\tau} \in \mathbb{R}_+^{m-1}$ and $\mathscr{F}$ be a holomorphic filtration of length m. A holomorphic filtration $\mathscr{F}$ is G-equivariantly $\boldsymbol{\tau}$-polystable if and only if there exists a smooth K-invariant hermitian metric h solution of the $\boldsymbol{\tau}$-Hermite-Einstein equation (2).

Proof See [2, Theorem 4.7].

Proposition 10 *Let $\mathscr{F}$ a holomorphic filtration over X of length m. Then $\mathscr{F}$ is G-equivariantly $\mathbf{R}$-Gieseker stable if and only if $Aut(\mathscr{F}) = \mathbb{C}$ and for k large, there exists a K-invariant metric $h_k \in Met(\mathcal{F} \otimes L^k)$ such that*

$$\widehat{\mathsf{B}}_{\mathcal{F}\otimes L^k, h_k} + \epsilon_k \sum_{j=1}^{m-1} \varepsilon_j \pi_{j,h_k}^{\mathscr{F}\otimes L^k} = \frac{N + \epsilon_k \sum_{j=1}^{m-1} \varepsilon_j r_j}{rV} Id_{\mathcal{F}\otimes L^k}$$

where

$$\epsilon_k = \frac{\chi(\mathcal{F} \otimes L^k)}{Vr - V\sum_{j>0} \varepsilon_j r_j}.$$

Proof First of all, since $\mathscr{F}$ is G-equivariantly $\mathbf{R}$-Gieseker stable it is in particular $\mathbf{R}$-Gieseker semistable and so one can consider for k sufficiently large a Gieseker space $\widetilde{\mathfrak{G}}_k$ as it is done in Section 2.2. The fact that the action lifts holomorphically implies that we have an action of G on $\widetilde{\mathfrak{G}}_k$. By definition of the embeddings $i_{k,j}$ (cf. p.14), the same holds for the space $\mathbf{\Pi}$ and therefore on the zero set of the $\mu_{\mathscr{F},k}$. Now by uniqueness, this implies that the balanced point in $\mathbf{\Pi}$ induced by the $i_{k,j}$ is K-invariant (the existence of this point is clear from Theorem 2). But this point only depends on the choice of the metric $H_k \in Met(H^0(\mathcal{F} \otimes L^k))$, and therefore the balanced metric is K-invariant. This gives the result using the Fubini-Study map FS_k.

With the previous results in hand, it is now clear that we have an equivariant version of Theorem 5, i.e that a K-equivariant $\boldsymbol{\tau}$-Hermite-Einstein metric can be approximated by K-invariant balanced metrics. Indeed, the balanced metrics that we construct in the proof of Theorem 5 are by uniqueness necessarily K-invariant.

Theorem 6 *Let $\mathscr{F}$ be an irreducible holomorphic filtration over X equipped with a K-equivariant $\boldsymbol{\tau}$-Hermite-Einstein metric h_{HE}. Then $\mathscr{F}$ is balanced and there exists a sequence of K-equivariant balanced metrics $\boldsymbol{h}_k$ which converges in C^∞ topology towards h_{HE} up to a conformal change.*

5.2 Filtrations, quivers and dimensional reduction

We now suppose that $X = M \times G/P$, and the action on M is trivial. We denote $p : X \to M$ and $q : X \to G/P$ the natural projections. By restriction, any G-equivariant vector bundle on X defines a P-equivariant holomorphic bundle on $M \times P/P \simeq M$. Conversely to any holomorphic P-equivariant bundle E on M, one can associate a G-equivariant bundle by considering the quotient of $G \times E$ by the action of $u \in P$ given by $u \cdot (g, e) = (g \cdot u^{-1}, u \cdot e)$ for which one has an action of $g' \in G$ by $g' \cdot (g, e) = (g'g, e)$. This principle of induction and restriction can also be applied to coherent sheaves. In fact, this equivalence of categories between G-equivariant holomorphic vector bundle on X and P-equivariant bundle on M can be extended in the following framework developped in [2,1].

Proposition 11 *Any coherent G-equivariant sheaf $\mathcal{F}$ on X admits a G-equivariant sheaf filtration*

$$\begin{aligned} &\mathscr{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow ... \hookrightarrow \mathcal{F}_m = \mathcal{F} \\ &\mathcal{F}_i/\mathcal{F}_{i-1} \cong p^*(\mathcal{E}_i) \otimes q^*(\mathcal{O}(\lambda_i)) \qquad 0 \leq i \leq m, \end{aligned}$$

where λ_i are increasing numbers and $\mathcal{E}_i$ are non zero coherent sheaves on M with trivial G-action. If $\mathcal{F}$ is a holomorphic vector bundle, then the $\mathcal{E}_i$ are also holomorphic vector bundles.

This motivates the following definitions (see [2,1] for details).

Definition 21 *A quiver is a pair of sets $\mathcal{Q} = \{\mathcal{Q}_v, \mathcal{Q}_a\}$ together with two maps $h, t : \mathcal{Q}_a \to \mathcal{Q}_v$. The elements of $\mathcal{Q}_v$ are called the vertices, and the elements of $\mathcal{Q}_a$ are called the arrows. For each $\overrightarrow{a} \in \mathcal{Q}_a$, the vertex $h\overrightarrow{a}$ is called the head of the arrow $\overrightarrow{a}$, and $t\overrightarrow{a}$ its tail. Moreover, $\mathcal{Q}$ will be locally finite, i.e we require $h^{-1}(v)$ and $t^{-1}(v)$ to be finite for all $v \in \mathcal{Q}_v$. A trivial path at $v \in \mathcal{Q}_v$ consists of the vertex v with no arrows. A non trivial path in $\mathcal{Q}$ is a sequence of arrows $p = \overrightarrow{a_0} \circ ... \circ \overrightarrow{a_l}$ that can be composed, i.e $t\overrightarrow{a_{i-1}} = h\overrightarrow{a_i}$. A relation of a quiver $\mathcal{Q}$ is a formal finite sum $r = c_1p_1 + ... + c_np_n$ of paths p_i with coefficients $c_i \in \mathbb{C}$. A quiver with relations $(\mathcal{Q}, \mathcal{R})$, is a pair consisting of a quiver $\mathcal{Q}$ and a set of relations $\mathcal{R}$ for $\mathcal{Q}$.*

Definition 22 *Let $\mathcal{Q}$ be a quiver. A $\mathcal{Q}$-sheaf $(\mathscr{E}, \boldsymbol{\phi})$ is a collection of $\mathscr{E}$ of coherent sheaves $\mathcal{E}_v$ for each $v \in \mathcal{Q}_v$ together with a collection of morphisms $\phi_{\overrightarrow{a}} : \mathcal{E}_{t\overrightarrow{a}} \to \mathcal{E}_{h\overrightarrow{a}}$ for each arrow $\overrightarrow{a} \in \mathcal{Q}_a$ such that $\mathcal{E}_v$ is zero for all but finitely many $v \in \mathcal{Q}_v$. A holomorphic $\mathcal{Q}$-bundle is a $\mathcal{Q}$-sheaf such that all the sheaves are holomorphic vector bundles. For a $\mathcal{Q}$-sheaf, any path in $\mathcal{Q}$ induces a morphism of sheaves and the trivial path induces the identity morphism $id : \mathcal{E}_v \to \mathcal{E}_v$. The $\mathcal{Q}$-sheaf $(\mathscr{E}, \boldsymbol{\phi})$ satifies a relation $r = c_1p_1 + ... + c_np_n$ if $\sum_i c_i\phi_{\overrightarrow{a_{i,0}}} \circ ... \circ \phi_{\overrightarrow{a_{i,l_i}}} = 0$ where l_i is the length of the path $p_i = \overrightarrow{a_{i,0}} \circ ... \circ \overrightarrow{a_{i,l_i}}$. Let $\mathcal{R}$ be a set of relations of $\mathcal{Q}$. A $(\mathcal{Q}, \mathcal{R})$-sheaf (resp. $(\mathcal{Q}, \mathcal{R})$-bundle) is a $\mathcal{Q}$-sheaf (resp. $\mathcal{Q}$-bundle) satisfying the relations $\mathcal{R}$.*

Remark 9 The notion of quiver is a natural generalisation of the notion of chain that appeared in [1]. A sheaf chain is a pair $\mathscr{C} = (\mathscr{E}, \boldsymbol{\phi})$ where $\mathscr{E} = (\mathcal{E}_0, ..., \mathcal{E}_m)$ is a $(m+1)$-tuple of coherent sheaves and a m-tuple $\boldsymbol{\phi} = (\phi_1, ..., \phi_m)$ of homomorphisms $\phi_i \in Hom(\mathcal{E}_i, \mathcal{E}_{i-1})$. We will later need the notion of a stability for a chain. If we denote the $\boldsymbol{\alpha}$-slope

$$\mu_{\boldsymbol{\alpha}}(\mathscr{C}) = \frac{\sum_{i=1}^m \deg(\mathcal{E}_i) - \sum_{i=1}^m \alpha_i rk(\mathcal{E}_i)}{\sum_{i=1}^m rk(\mathcal{E}_i)}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m)$ is a collection of real numbers, then the chain $\mathscr{C}$ is called $\boldsymbol{\alpha}$-stable if $\mu_{\boldsymbol{\alpha}}(\mathscr{C}') < \mu_{\boldsymbol{\alpha}}(\mathscr{C})$ for all proper subchains $\mathscr{C}'$ of $\mathscr{C}$.

We have the following theorem from [2, Theorem 2.5].

Theorem 5.21 There exists a one-to-one correspondence between the categories of G-equivariant holomorphic vector bundles on $X = M \times G/P$ and of holomorphic $(\mathcal{Q}, \mathcal{R})$-bundles on M.

Let's give a simple example. We have a correspondence at the level of holomorphic objects between the extensions on X of the form

$$0 \to p^*\mathcal{E}_0 \to E \to p^*\mathcal{E}_1 \otimes q^*\mathcal{O}(2) \to 0$$

and the triple $(\mathcal{E}_0, \mathcal{E}_1, \phi_1)$ where $\phi_1 \in Hom(\mathcal{E}_1, \mathcal{E}_0)$. Indeed, Kunneth formula gives us $H^1(X, p^*(\mathcal{E}_0 \otimes \mathcal{E}_1^*) \otimes q^*\mathcal{O}(-2)) \simeq H^0(M, \mathcal{E}_0 \otimes \mathcal{E}_1^*) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2))$ and if we now fix an element in $H^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq \mathbb{C}$, the homomorphism ϕ_1 can be identified with the extension class defining E.

5.3 Dimensional reduction and applications

Let $\omega' = p^*\omega + q^*\omega_\epsilon$ (where ω_ϵ is the K-invariant smooth Kähler form constructed in [2, Lemma 4.8]) be a Kähler form on $X = M \times G/P$. We have the following central theorem [2, Theorem 4.13] of dimensional reduction between quivers and $\boldsymbol{\tau}$-Hermite-Einstein filtrations.

Theorem 5.31 Let $\mathcal{F}$ be a G-equivariant vector bundle on X, and let $\mathscr{F}$ be a G-equivariant filtration associated to $\mathcal{F}$ of length $(m+1)$. Let $(\mathscr{E}, \boldsymbol{\phi})$ the corresponding holomorphic $(\mathcal{Q},\mathcal{R})$-bundle on M. Then $\mathscr{F}$ admits a K-invariant $\boldsymbol{\tau}$-Hermite-Einstein metric with respect to ω' if and only if for each $v \in \mathcal{Q}_v$ such that the holomorphic vector bundle $\mathcal{E}_v$ of $(\mathscr{E}, \boldsymbol{\phi})$ is non trivial, there exists smooth hermitian metrics $h_v \in Met(\mathcal{E}_v)$ satisfying

$$\sqrt{-1}n_v \Lambda F_{h_v} + \sum_{\overrightarrow{a} \in h^{-1}(v)} \phi_{\overrightarrow{a}} \circ \phi_{\overrightarrow{a}}^{*h_v} - \sum_{\overrightarrow{a} \in t^{-1}(v)} \phi_{\overrightarrow{a}}^{*h_v} \circ \phi_{\overrightarrow{a}} = \tau'_v Id_{\mathcal{E}_v} \tag{38}$$

where $n_v = \dim(\mathrm{M_v})$ is the multiplicity of the irreducible representation $\mathrm{M_v}$ of P attached to v.

The relation between the real positive numbers (τ_i) and (τ'_v) is made clear in [2, Section 4.2.2]. We will refer to the system (38) as a quiver Vortex equation. Many equations from litterature can be obtained as a particular case of a quiver Vortex equation. We shall now give the main examples of such equations.

– The case of coupled Vortex equations. Indeed, consider $X = M \times \mathbb{P}^1$ and the group action $SL(2) = SL(2,\mathbb{C})$ given by the trivial action over M and the standard action on $\mathbb{P}^1$ via the natural identification

$$\mathbb{P}^1 = SL(2)/\mathfrak{P},$$

where $\mathfrak{P}$ stands for the parabolic subgroup of lower triangular matrices of $SL(2)$. In that case studied in [1], the previous theorem can be rephrased in the following way. A holomorphic filtration $\mathscr{F}$ on X admits a $SU(2)$-invariant $\boldsymbol{\tau}$-Hermite-Einstein metric respectively to $p^*\omega + q^*\omega_{FS}$ (where ω_{FS} denotes the Fubini-Study metric on $\mathbb{P}^1$) if and only if the chain $\mathscr{C} = (\mathscr{E}, \boldsymbol{\phi})$ admits a $(m+1)$-tuple of hermitian metrics $\mathbf{h} = (h_0, ..., h_m)$ satisfying the following chain of Vortex equations (also called coupled Vortex equations),

$$\begin{gathered}
\sqrt{-1}\Lambda F_{h_0} + \frac{1}{2}\phi_1 \circ \phi_1^{*h_0} = \tau_0 Id_{\mathcal{E}_0}, \\
\sqrt{-1}\Lambda F_{h_i} + \frac{1}{2}\left(\phi_{i+1} \circ \phi_{i+1}^{*h_i} - \phi_i^{*h_i} \circ \phi_i\right) = (\tau_i - 2i) Id_{\mathcal{E}_i} \quad (1 \leq i \leq m-1), \\
\sqrt{-1}\Lambda F_{h_m} - \frac{1}{2}\phi_m^{*h_m} \circ \phi_m = (\tau_m - 2m) Id_{\mathcal{E}_m}.
\end{gathered} \tag{39}$$

Note that these equations have been related to the computation of some twisted Gromov-Witten invariants [33]. It covers the interesting case of holomorphic triples (E_1, E_2, ϕ) where $\phi \in H^0(X, Hom(E_1, E_2))$ which is simply a holomorphic chain of length 1. The coupled Vortex equations relative to these triples have been extensively studied (see for instance [19,10,9]).

- The case of Hitchin's self-duality equation [22] over a complex curve C,

$$F_h^{\perp} + [\Phi, \Phi^{*_h}] = 0, \tag{40}$$

where $\Phi \in H^0(M, End(E) \otimes K_C)$ and $F_h^{\perp}$ is the trace free part of the curvature F_h. The notion of stability considered here is simply the usual one for the holomorphic vector bundles restricted to the Φ-invariant subbundles. On the moduli space $\mathcal{M}$ of stable Higgs bundles of degree 0 over C, we have the so-called S^1 action of Hitchin,

$$g \cdot (A, \Phi) = (A, g\Phi) \in \mathcal{A}(E) \times H^0(M, End(E) \otimes K_C)$$

which preserves the natural Kähler form on $\mathcal{M}$. A stable (E, A, Φ) bundle represents a fixed point of this action if and only if there exists a Gauge transformation ϑ such that $D_A\vartheta = 0$ and $[\vartheta, \Psi] = \sqrt{-1}\Psi$ (cf. [23,36]). Then, the Higgs bundle E is called *critical* and can be decomposed holomorphically

$$E = \bigoplus_{i=0}^{d} E_i,$$

and ϑ acts with increasing weights $\lambda_i \in \mathbb{R}$ on each factor E_i, i.e one has a variation of Hodge structure. We also have some non trivial morphims $\Phi_i : E_i \to E_{i+1} \otimes K_C$ with $\Phi = \bigoplus_i \Phi_i$. If one denotes now $\mathcal{E}_i = E_{d-i} \otimes K_C^{d-i}$, then we can define a holomorphic chain $(\mathscr{E}, \boldsymbol{\phi})$ by considering the morphisms

$$\phi_i = \Phi_{d-i} \otimes Id : \mathcal{E}_i \to \mathcal{E}_{i-1}.$$

Now, to find a solution $h \in Met(E)$ of (40) for a critical Higgs bundle over C is equivalent to find for $0 \leq i \leq d$ smooth metrics $h_i \in Met(\mathcal{E}_i)$ solutions of the couplex Vortex equations associated to the holomorphic chain $(\mathscr{E}, \boldsymbol{\phi})$. Indeed, this chain is $\boldsymbol{\alpha}$-stable with respect to the weight $\boldsymbol{\alpha} = ((m-i)\deg(K))_{i=0,..,d}$ and up to a conformal change e^{ρ} given by the potential of the fixed metric on K_C, we have $h = \bigoplus_{i=0}^{d} e^{-(d-i)\rho} h_i$.

- The case of Witten triples (Vafa-Witten equations studied in [40]). Let L be a line bundle on a complex surface S and $(\mathcal{L}, \phi, \theta)$ a triple formed by a holomorphic structure $\mathcal{L}$ on L, a holomorphic section $\phi \in H^0(M, \mathcal{L})$ and a morphism $\theta : \mathcal{L} \to K_S$. The triple $(\mathcal{L}, \phi, \theta)$ is called β-stable if $\deg(\mathcal{L}) < \beta$ and $\phi \neq 0$ or $\beta < \deg(\mathcal{L})$ and $\theta \neq 0$. A triple is β-stable if and only if $(\phi, \theta) \neq (0,0)$ and there exists a metric h on L satisfying the equation

$$\sqrt{-1}\Lambda F_h + \frac{1}{2}(|\phi|_h^2 - |\theta|_h^2) = \beta. \tag{41}$$

Suppose now that $\deg(\mathcal{L}) < \beta$ and ϕ does not vanish (we will say that the triple is *special*), then we can consider the chain

$$((K_S, \mathcal{L}, \mathcal{O}), (\theta, \phi))$$

which is $\boldsymbol{\alpha}$-stable with $\boldsymbol{\alpha} = (\alpha_0, \beta, \alpha_2)$ if $\deg(\mathcal{L}) < \beta + \deg(K_S) - \alpha_0$. If we choose $\alpha_0 - \deg(K_S) > 0$ sufficiently small, then we know the existence of solutions for the associated couplex Vortex equations and in particular there exist two metrics $h_0 \in Met(K_S)$ and $h_1 \in Met(L)$ such that

$$\sqrt{-1}\Lambda F_{h_0} + \frac{1}{2}|\theta|^2 = \alpha_0, \tag{42}$$

$$\sqrt{-1}\Lambda F_{h_1} + \frac{1}{2}\left(|\phi|^2 - |\theta|^2\right) = \beta. \tag{43}$$

Note that here the norms $|\theta|^2$ (resp. $|\phi|^2$) are computed with respect to both metrics h_0 and h_1 (resp. h_1 and a trivialisation frame on the structure sheaf). For $\alpha_0 - \deg(K_S)$ sufficiently small, the term $A = |\phi|^2 - |\theta|^2$ is going to be positive by (42) and we can recover (41) from (43) if we do a conformal change. Indeed, this leads us to find a smooth function f on S such that $\Delta f + Ae^f = B$ with $\int_S BdV = \beta - \deg(\mathcal{L})$. But from the work of Kazdan and Warner [27], the existence and uniqueness of f holds as soon as $A > 0$ and $\int_S BdV > 0$.

– The case of Bradlow pairs [8] studied in [19]. If (E, ϕ) is a pair (i.e. E is a holomorphic vector bundle and $\phi \in H^0(M, E)$) on (M, ω) and if F is given by extension on $X = M \times \mathbb{P}^1$

$$0 \to p^*E \to F \to q^*\mathcal{O}(2) \to 0$$

then (E, ϕ) is λ-stable in the sense of Bradlow if and only if F is Mumford-stable with respect to the polarization associated to

$$p^*\omega + \frac{2V}{(r(E)+1)\lambda - \deg(E)} q^*\omega_{FS}.$$

Of course, there is a natural identification between the pair (E, ϕ) and the triple $(E, \mathcal{O}, \phi)$. Nevertheless, except in the case when $rk(E) = 1$, it is not possible in general to relate in a simple way Bradlow's equation [8] to equation (38). Instead, one obtains a new equation, called *almost* Vortex equation

$$\sqrt{-1}\Lambda F_h + e^{-u}\phi \otimes \phi^{*_h} = \lambda Id_E,$$

where u depends on the choice of a trivialisation on the structure sheaf. Even if this metric h and the solution of Bradlow's equation are not related by a conformal change, they define the same point in the moduli space of solutions [11].

Now, suppose that we consider a metric (or a family of metrics) solution to one of the previous equations (38),(39),(40),(41). Thanks to Theorem 5.31, we know that these solutions are related to a K-invariant $\boldsymbol{\tau}$-Hermite-Einstein metric for a filtration on $X = M \times G/P$. From the equivariant version of the Approximation theorem (Theorem 6), we see that these solutions can be approximated by 'algebraic metrics', i.e metrics coming from a G.I.T construction as a zero of certain moment maps in finite dimensional setting. Finally we get,

Theorem 7 *Let $(\mathscr{E}, \boldsymbol{\phi})$ an irreducible $(\mathcal{Q}, \mathcal{R})$-bundle on a smooth projective manifold M such that for each $v \in \mathcal{Q}_v$ with $\mathcal{E}_v$ non trivial, there exists a smooth hermitian metrics $h_v \in Met(\mathcal{E}_v)$ satisfying the quiver Vortex equation (38). Then, up to some renormalizations by conformal changes, every metric h_v is the limit in C^∞ topology of a sequence of algebraic metrics. In particular, the solutions of coupled Vortex equations, critical Hitchin's self-duality equations over a curve, special Vafa-Witten equations can be approximated by algebraic metrics. For an irreducible stable pair, the sequence of algebraic metrics obtained converges to the solution of an almost Vortex equation.*

6 Appendix

6.1 Endomorphism $\Pi_h^{\mathscr{F},\boldsymbol{\tau}}$

In this part, we gather some elementary and technical results which are used in Section 4.

Let $\mathscr{F}$ be a holomorphic filtration of length m, h a hermitian metric on $\mathcal{F}$ and $\pi_{h,i}^{\mathscr{F}}$ the h-orthogonal projection onto the bundle $\mathcal{F}_i \subset \mathcal{F}$. For two smooth hermitian metrics h_1 and h_2 on $\mathcal{F}$, we know that they are related by the existence of an endomorphism η such that

$$h_1(X,Y) = h_2(\eta X, Y)$$

and η is hermitian with respect to h_2 and definite positive. In particular, it is well known that $F_{h_1} = F_{h_2} + \overline{\partial}(\eta^{-1}\partial_{h_2}\eta)$.

Notation We set by $h \cdot \eta$ the metric $h(\eta \cdot, \cdot)$ for $\eta \in End(\mathcal{F})$, hermitian with respect to h.

Lemma 15 *For all $(m-1)$-tuple $\{\tau_1, .., \tau_{m-1}\}$ of real numbers and all h-hermitian endomorphism $\boldsymbol{\eta} \in End(\mathcal{F})$ such that $h' = h \cdot \boldsymbol{\eta}$, we have*

$$d\left(\sum_{i=1}^{m-1} \tau_i \pi_{h',i}^{\mathscr{F}}\right) = \sum_{i=1}^{m-1} \tau_i \pi_{h,i}^{\mathscr{F}} d\boldsymbol{\eta}\left(Id - \pi_{h,i}^{\mathscr{F}}\right).$$

Proof First of all, we can restrict to one of the factors $\pi_{h,i}^{\mathscr{F}}$, h-orthogonal projection onto the subbundle $\mathcal{F}_i$. Let $t \mapsto \pi_i(t)$ be a one parameter family of projections onto the bundle $\mathcal{F}_i$ such that $\pi_i(0) = \pi_{h,i}^{\mathscr{F}}$. Since we have the relations

$$\pi_i(t)\pi_i(t) = \pi_i(t), \qquad \pi_i(0)\pi_i(t) = \pi_i(t),$$

we obtain that

$$\pi_i(0)\pi_i(0)' = \pi_i(0)',$$

i.e. $\mathrm{Im}(\pi_i(0)') \subset \mathcal{F}_i$. From another side, $\pi_i(0)'\pi_i(0) = \pi_i(0)' - \pi_i(0)\pi_i(0)' = 0$ and therefore $\ker(\pi_i(0)') \supset \mathcal{F}_i$. Thus,

$$\pi_i(0)' = \pi_i(0)\pi_i(0)'(Id - \pi_i(0)),$$

and the space of solutions of this equation is $Hom(\mathcal{F}_i^\perp, \mathcal{F}_i)$. We notice that the differential is necessarily $U(\mathcal{F}_i^\perp)\times U(\mathcal{F}_i)$ invariant. Finally, we can apply Schur lemma since $U(\mathcal{F}_i^\perp)\times U(\mathcal{F}_i)$ acts irreducibly. Then, up to a multiplicative constant, the differential is given by

$$X \mapsto \pi_i(0)X(Id - \pi_i(0)).$$

Set $h_t = h\cdot(Id+\eta_t)$ avec $\eta_0 = 0$ and choose a h-orthonormal basis $(e_j^0)_{j=1,..,r}$ for which the first r_i vectors generate $\mathcal{F}_i$. Then the new h_t-orthonormal basis $(e_j^t)_{j=1,..,r}$ is given by

$$R(e_j^t) = (e_i^0), \tag{44}$$

where R is the unique upper triangular matrix with positive diagonal coefficients that satisfies the relation $RR^{*h_t} = Id + \eta_t$. Now, by differentiating (44) at $t = 0$, we have for $j \le r_i$,

$$\left(de_j^t\right)_{t=0} = -\frac{1}{2} d\left(\eta_0\right)_{jj} e_j^0 - \sum_{k<j} d\left(\eta_0\right)_{jk} e_k^0.$$

Thus, differentiating h_t at $t=0$, we have

$$d\left(\sum_{j=1}^{r_i} e_j^t \otimes e_j^{t*h_t}\right)_{t=0} = \sum_{j=1}^{r_i} e_j^0 \otimes e_j^{0*h_0} d\eta_0 - \sum_{j=1}^{r_i} e_j^0 \otimes e_j^{0*h_0} d\eta_0 \sum_{k=1}^{r_i} e_k^0 \otimes e_k^{0*h_0},$$

which allows us to conclude. □

Now, by direct application of the previous lemma we have,

Lemma 16 *For all $(m-1)$-tuple $\{\tau_1, ...\tau_{m-1}\}$ of real numbers and all hermitian endomorphism $\boldsymbol{\eta} \in End(\mathcal{F})$, we have*

$$\sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h\cdot(Id+\boldsymbol{\eta}),i} = \sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h,i} + \varPi_h^{\mathscr{F},\boldsymbol{\tau}}(\boldsymbol{\eta}) + \mathbf{O}(\boldsymbol{\eta}^2)$$

where we have fixed the endomorphism,

$$\varPi_h^{\mathscr{F},\boldsymbol{\tau}} : \boldsymbol{\eta} \mapsto \sum_{i=1}^{m-1} \tau_i \pi^{\mathscr{F}}_{h,i} \boldsymbol{\eta} \left(Id - \pi^{\mathscr{F}}_{h,i}\right).$$

Here $\mathbf{O}(\boldsymbol{\eta}^2)$ represents an hermitian endomorphism such that its Hilbert-Schmidt norm can be bounded by $O(\|\boldsymbol{\eta}\|^2_{C^0})$.

6.2 Resolution of a certain elliptic equation

We will need the following classical Kähler identities.

Lemma 17 *Let E be a hermitian holomorphic vector bundle E over a Kähler manifold and F_E be its Chern curvature. We have the commuting identities:*

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*, \qquad [\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*, \qquad \Delta_{\bar{\partial}} = \Delta_{\partial} + [\sqrt{-1}F_E, \Lambda].$$

In all the following, we will assume that the $\{\tau_1, ..., \tau_m\}$ are non negative.

Lemma 18 *Let $\mathscr{F}$ be a simple holomorphic filtration over M and let $\Psi : End(\mathcal{F}) \to End(\mathcal{F})$ be a positive self-adjoint operator of order zero. Then, for all hermitian metric $h \in Met(\mathcal{F})$, it is always possible to find a smooth solution, which preserves the filtration, of the following elliptic system:*

$$\Lambda_\omega \overline{\partial}\partial \mathsf{Q}' + \Psi(\mathsf{Q}') = \mathsf{Q}$$

for all smooth endomorphism Q such that $\mathsf{Q}(\mathcal{F}_i) \subset \mathcal{F}_i$ and $\int_M \mathrm{tr}(\mathsf{Q})dV = 0$.
Moreover, if $\mathscr{F}$ is a holomorphic filtration such that there exists $\mathcal{F}$ a conformally $\boldsymbol{\tau}$-Hermite-Einstein metric h, and if one has fixed

$$\Psi : U \mapsto \Pi_h^{\mathscr{F}, \boldsymbol{\tau}}(U), \tag{45}$$

then Q is self-adjoint if and only if Q' is self-adjoint.

Proof We need to see that the operator $\Lambda_\omega \overline{\partial}\partial + \Psi$ has trivial kernel. Recall that this operator is elliptic (of order 2) positive and self-adjoint since the τ_i are positive. Let us consider the kernel of this operator: $(\partial^*\partial)U = 0$ implies $|\partial U|_h = 0$, i.e. since $\mathscr{F}$ is simple, $U = \gamma Id$ with γ constant on M. If $Id \in \ker \Psi$ then, by Fredholm alternative, the elliptic system admits a solution if $\langle Id, Q\rangle = \int_M \mathrm{tr}(\mathsf{Q})dV = 0$. The uniqueness is obvious once it has been assumed that $\int_M \mathrm{tr}(\mathsf{Q}')dV = 0$. If $Id \notin \ker \Psi$, then the system admits a unique solution. The defined operator by (45) is self-adjoint and positive. Moreover, again with Kähler identities,

$$\begin{aligned}
\sqrt{-1}\Lambda\bar{\partial}\partial \mathsf{Q}'^* &= \Delta_\partial \mathsf{Q}'^*, \\
&= -\left(\Delta_{\bar{\partial}} \mathsf{Q}'\right)^*, \\
&= \left(\Delta_\partial \mathsf{Q}' - [\sqrt{-1}\Lambda F_h, \mathsf{Q}']\right)^*, \\
&= \left(\Delta_\partial \mathsf{Q}' - [\textstyle\sum_i \pi^{\mathscr{F}}_{h,i}, \mathsf{Q}']\right)^*,
\end{aligned}$$

and we get

$$\left(\sqrt{-1}\Lambda\left(\overline{\partial}\partial \mathsf{Q}'\right) + \sum_i \pi^{\mathscr{F}}_{h,i}\mathsf{Q}'\right)^* = \sqrt{-1}\Lambda\left(\overline{\partial}\partial \mathsf{Q}'^*\right) + \sum_i \pi^{\mathscr{F}}_{h,i}\mathsf{Q}'^*.$$

Hence,

$$\left(\sqrt{-1}\Lambda\left(\overline{\partial}\partial \mathsf{Q}'\right) + \Psi(\mathsf{Q}')\right)^* = \sqrt{-1}\Lambda\left(\overline{\partial}\partial \mathsf{Q}'^*\right) + \Psi(\mathsf{Q}'^*),$$

and by uniqueness of the solution, we have that Q' is hermitian and definite positive if and only if it is the case for Q. $\square$

Acknowledgements The author is particularly grateful to Philippe Eyssidieux for introducing him to the subject. He deeply thanks Richard Thomas for enlightening conversations and Eva Miranda for her constant encouragement and support.

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